Performance Evaluation of Fragmented Structures: A Theoretical Study

Ed Coffman¹, Robert Margolies¹, Peter Winkler², Gil Zussman¹
¹Columbia University, New York, NY
²Dartmouth College, Hanover, NH
Motivation

Hard and Solid-state Disk Drives

OFDMA, LTE, and Cognitive Radio
Motivation

Hard and Solid-state Disk Drives
- Provides interleaving of data allowing for faster read times in some cases
- Disk head motion significantly increases write times

OFDMA, LTE, and Cognitive Radio
- Provides frequency diversity
- Requires complex hardware solutions
Related Work

• Solid-state Disk Drives and OFDMA

• Classical Fragmentation Problems
Outline

✓ Motivation
  • Model + Example
  • Motivating Numerical Results
  • Asymptotic Theory of Complete Fragmentation
    o Case 1: Items of size 1 or 2
    o Case 2: Items up to size K (size 1 items have positive probability)
    o Case 3: Items up to size K

• Convergence to Complete Fragmentation
• Conclusions
Model

- Resource is modeled as a sequence of $M > 1$ slots.
- FIFO queue under full load: there are always waiting items.
- Item sizes are i.i.d. with distribution $q = \{q_1, \ldots, q_K\}$ and have independent i.i.d. exponential residence times.
- An allocation algorithm allocates available gaps in the resource to waiting items.
Unbounded queue of waiting requests

Allocation Algorithm

Next-fit: Requests served in order using left-to-right scans starting where the previous scan left-off

Resource, \( M=20 \) slots

6 available slots
Model

Unbounded queue of waiting requests

Allocation Algorithm

Next-fit: Requests served in order using left-to-right scans starting where the previous scan left-off

Resource, $M=20$ slots

Fragment 1: 6 slots
Complete Fragmentation

After a $long$ time, does fragmentation progress to a point where nearly all items are completely fragmented?

Complete fragmentation: When no two allocated slots of an item are adjacent.
Numerical Examples of Fragmentation Allocation

- Unbounded queue restricted to waiting size-1 or size-2 items
  - $P(\text{size-1 item}) = q_1 = \frac{1}{2}$
  - $P(\text{size-2 item}) = q_2 = \frac{1}{2}$

<table>
<thead>
<tr>
<th>Allocation Algorithm</th>
<th>Average # of Unfragmented Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = 10 slots</td>
<td>0.79</td>
</tr>
<tr>
<td>M = 100 slots</td>
<td>0.89</td>
</tr>
<tr>
<td>M = 1,000 slots</td>
<td>0.91</td>
</tr>
<tr>
<td>M = 10,000 slots</td>
<td>0.92</td>
</tr>
</tbody>
</table>
Numerical Examples of Fragmentation

Item sizes Uniform on

- \{1,2\}
- \{1,2, \ldots, 5\}
- \{1,2, \ldots, 10\}

Probability of a size-\(j\) item allocated \(j\) fragmented slots.

Nearly all items are completely fragmented as \(M \to \infty\)
Motivation

Model + Example

Motivating Numerical Results

Asymptotic Theory of Complete Fragmentation
  - Case 1: Items of size 1 or 2
  - Case 2: Items up to size K (size 1 items have positive probability)
  - Case 3: Items up to size K

Convergence to Complete Fragmentation

Conclusions
Complete Fragmentation: Case 1

- Infinite queue restricted to waiting size-1 or size-2 items
- \( P(\text{size-1 item}) = q_1 > 0 \)
- \( P(\text{size-2 item}) = q_2 > 0 \)
- Stable state represented as \((G,H)\) with \(G < H\)
  - \(G\) is size of available gaps
  - \(H\) is size of Head-of-line item

Goal: Show that as the size of the resource \((M)\) grows, *nearly* all of the size-2 items are *fragmented*. 

![Resource Allocation Diagram]
Complete Fragmentation: Case 1

- Infinite queue restricted to waiting size-1 or size-2 items
- \( P(\text{size-1 item}) = q_1 > 0 \)
- \( P(\text{size-2 item}) = q_2 > 0 \)
- Stable state represented as \((G,H)\) with \(G < H\)
  - \(G\) is size of available gaps
  - \(H\) is size of Head-of-line item

\[
\begin{align*}
\pi_{0,1} &= \frac{q_1}{2 - q_1} \\
\pi_{0,2} &= \frac{1 - q_1}{2 - q_1} \\
\pi_{1,2} &= \frac{1 - q_1}{2 - q_1}
\end{align*}
\]

Balance the rate of increase/decrease of the number of unfragmented size 2 items, denoted \(N_{2,1}\)

Complete Fragmentation: Case 1

Event – size 1 item departs with probability $q_1$
Result: No change in $N_{2,1}$ and move to state (1,2)

Event – size 2 item departs with probability $q_2$
Result: Waiting size-2 item will occupy departed size 2 item and no change in $N_{2,1}$

Conclusion: $N_{2,1}$ does not change in state (0,2)
Complete Fragmentation: Case 1

Event – size 1 item departs with probability $q_1$
Result: $N_{2,1}$ increases at rate 1 if a size-1 item departs adjacent to the present gap

Event – size 2 item departs with probability $q_2$
Result: $N_{2,1}$ increases at rate at most 2 if one of the 2 slots in the departing size-2 item is adjacent to the present gap.

Conclusion: $N_{2,1}$ increases at rate at most 2 when in state (1,2)
Complete Fragmentation: Case 1

<table>
<thead>
<tr>
<th>Allocation Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 slots</td>
</tr>
<tr>
<td>2 slots</td>
</tr>
<tr>
<td>1 slot</td>
</tr>
<tr>
<td>2 slots</td>
</tr>
<tr>
<td>1 slot</td>
</tr>
</tbody>
</table>

Event – size 1 item departs with probability $q_1$
Result: No change in $N_{21}$ and move to state $(0,1)$ or $(0,2)$

Event – size 2 item departs with probability $q_2$
Result: $N_{21}$ decreases by 1 if an unfragmented size-2 item departs which occurs with rate $N_{21}(t)$

Conclusion: $N_{21}$ decreases at rate $N_{21}(t)$ when in state $(0,1)$
Complete Fragmentation: Case 1

All but a constant number of size-2 items become completely fragmented.

Resource-size grows to infinity.

\[ \pi_{0,1} = \frac{q_1}{2 - q_1} \]

Decreases at rate \( N_{21}(t) \)

\[ \pi_{0,2} = \frac{1 - q_1}{2 - q_1} \]

No Change in \( N_{21} \)

\[ \mathbb{E}[N_{21}] \leq 2 \frac{1 - q_1}{q_1} \]

\[ \pi_{1,2} = \frac{1 - q_1}{2 - q_1} \]

\( N_{21} \) Increases at rate at most 2

\[ \leq 2(1 - q_1)/q_1 \]
Complete Fragmentation: Case 2

- Infinite queue of waiting items up to size-$K$
- $P(\text{item of size } j) = q_j$
- Positive probability of size-1 items: $q_1 > 0$

All but a constant number of items become completely fragmented

Resource-size grows to infinity

Allocation Algorithm

non-completely fragmented items
Complete Fragmentation: Case 3

- Infinite queue of waiting items up to size-$K$
- $P(\text{item of size } j) = q_j$
- Positive probability of size-1 items: $q_1 > 0$
- Item sizes with positive probability have a non-trivial common divisor

The fraction of completely fragmented items tends to 1 as resource size grows to infinity.
# Complete Fragmentation

<table>
<thead>
<tr>
<th>Case</th>
<th>Number of unfragmented items:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: Items of Size 1 and 2</td>
<td>Bounded by $\mathbb{E}[N_{21}] \leq 2 \frac{1 - q_1}{q_1}$</td>
</tr>
<tr>
<td>Case 2: Items up to size $K$ (size 1 items have positive probability)</td>
<td>Bounded by a constant $C$</td>
</tr>
<tr>
<td>Case 3: Items up to size $K$</td>
<td>Grow at a rate $o(M)$</td>
</tr>
</tbody>
</table>

Implication: For all cases, complete fragmentation is approached as

$$\lim_{M \to \infty} \frac{\# \text{ of Unfragmented Items}}{M} = 0$$

where $M$ is the size of the resource.
Outline

- Motivation
- Model + Example
- Initial Experimental Results
- Asymptotic Theory of Complete Fragmentation
  - Case 1: Items of size 1 or 2
  - Case 2: Items up to size $K$ (size 1 items w.p.p)
  - Case 3: Items up to size $K$

- Convergence to Complete Fragmentation
- Conclusions
Convergence to Nearly Complete Fragmentation: Time

Fragmentation growth is most logarithmic in the number of departures.

When items are of size 1 or 2, the number of departures until the fraction of fragmented requests increase by $\gamma$ is

$$M \frac{1 - q_1}{2 - q_1} \ln \frac{1}{1 - \gamma}$$

$M = 100,000$ slots
Item size distribution $\sim$ uniform on $\{1, \ldots, 5\}$
Convergence to Nearly Complete Fragmentation: Space

Recall that: \( \lim_{M \to \infty} \frac{\text{# of Unfragmented Items}}{M} = 0 \)

The size of the resource \( M \) required to approach complete fragmentation can be enormous even for simple item size distributions.

Item size distribution ~ \{1,9\} with probabilities \( q_1 \) and \( q_9=(1-q_1) \)
Conclusions

- Nearly all items become completely fragmented in statistical equilibrium when the resource size grows to infinity
  - Proofs for cases 1 and 2 balance the rates at which the number of non-fragmented items increases and decreases in equilibrium
  - Good news: frequency diversity in OFDMA, defragmentation software
  - Bad news: requires complex hardware solutions

- Convergence rates can be surprisingly slow in the limits of time and resource size

- Robert Margolies
- robm@ee.columbia.edu