Quantifying the Effect of k-line Failures in Power Grids

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***Abstract*—In this paper, we quantify the effect of k-line failures on flow changes using the DC power flow model. We demonstrate that our approach results in efficient tools that can be used to reduce both the total number of cases needed to be analyzed and the computational complexity in power grid contingency analysis. After providing an analytical update of the pseudo-inverse of the admittance matrix following a k-line failure, we compute the k-*line outage distribution matrix*. The k-line outage distribution matrix is a generalization to the line outage distribution factor for single line failures. We obtain a matrix equation based on the submatrices of the *matrix of equivalent reactance values*, relating changes in power flows to the initial flows on the failed lines. We also define and analytically compute the *disturbance value* of a failure – the weighted sum of squares of the flow changes – and show that it can be computed for any set of failures in $O(1)$ independent of the size of the power grid. Finally, we numerically compute disturbance values for all possible choices of 3-line failures in IEEE 118-bus and 300-bus systems and show that the disturbance values provide a clear separation between failures with higher impact and lower impact.**

I. Introduction

Recent large scale power grid blackouts in Turkey (2015), India (2013), and the U.S. (2003) exposed the insufficiency of current control tools to protect the grid against cascading failures. A tremendous amount of research in the past decade has focused on resolving this problem by developing monitoring and control techniques to make power grids more robust and secure. The problem of contingency identification is one of the important research directions focusing on improving the reliability of power grids [1], [2], [3], [4], [5], [6], [7], [8]. A specific problem is how to detect all possible failure events that can hinder the regular operation of the grid.

Many great ideas have been developed for contingency analysis in power grids since the advent of the modern power transmission network. *Current injection* methods were used to analyze the effect of line failures in [2], [5], [8], [9]. In particular, [9] introduced the notion of *line outage distribution factors* that inspired many other studies including the work presented in this paper. In [1] and [10], multiple matrix updates were used to study the effect of single line failures and to speed up the computation of the power flows after line failures. [7] used matrix updates to study the effect of two line failures and used the results to introduce an algorithm for the $N-2$ contingency problem. More optimization-based techniques for contingency analysis of the grids were explored in [3], [4]. A mixed-integer model for the $N-k$ contingency problem was presented and used in [3], while [4] focused on identifying the most probable failure modes in static load distribution using a linear-program. In a more recent innovative paper, probabilistic algorithms were developed to identify collections of multiple contingencies that initiate cascading failure [11].

In this paper, we build on the result of [10] that focused on single line failure events and *provide a metric for quantifying the effect of k-line failures on flow changes*. Similar to the most of the previous work on contingency analysis, we use the linearized DC approximation of the power flows, due to the complexities associated with the AC power flow model [12]. First, we present an analytical update of the pseudo-inverse of the admittance matrix after a k-line failure event. While our approach is similar to [13], we use the pseudo-inverse instead of the truncated inverse of the admittance matrix. Using this result, we define and analytically compute the *k-line outage distribution matrix* which generalizes the definition of the line outage distribution factors for single line failures [9].

While a k-line outage distribution matrix captures all effects of a k-line failure on flow changes, it is not efficient to compute and store this matrix for contingency analysis in large power grids. To overcome this problem, we use the *matrix of equivalent reactance values* originally defined and used in [10] to efficiently compute the sum of changes in the power flows after k-line failures and to provide a metric that captures the essence of flow changes after failures. In particular, we provide a matrix equation based solely on the submatrices of the matrix of equivalent reactance values, relating the changes in power flows to the initial flows on the failed lines. Since the matrix of equivalent reactance values needs to be computed only once, this matrix equation can be written for any k-line failures without further computations. We also define and analytically compute the *disturbance value* of a failure (the weighted sum of squares of the flow changes) and show that this computation can be done for a k-line failure in $O(1)$ as long as k is much smaller than the total number of lines, the case in contingency analysis of the power grids. Hence, the effect of k-line failures can be computed independent of the size of the grid using our metric and results.

To show that disturbance values provide a clear separation between failures with higher impact and lower impact, we compute the disturbance values for all possible choices of 3-line failures in the IEEE 118-bus and 300-bus systems. We demonstrate that by ranking cases based on their disturbance values and considering only cases with high disturbance value, we are able to decrease the total number of cases needed to be analyzed for contingency analysis by more than 96%.

The remainder of this paper is organized as follows. Section II describes the model and provides definitions. In Section III, we analyze the effect of k-line failures and provide results to quantify these effects. Section IV presents numerical results, and Section V provides concluding remarks and suggestions for future work. Some proofs and lemmas are moved to the Appendix to make the text easier to read.

II. Model and Definitions

A. DC Power Flow Model

We adopt the linearized (or DC) power flow model, which is widely used as an approximation for the non-linear AC power flow model [14], [15]. We represent the power grid by a connected directed graph $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ and $E = \{e_1, \ldots, e_m\}$ are the set of nodes and edges corresponding to the buses and transmission lines, respectively (the definition implies $|V| = n$ and $|E| = m$). Each edge e is a set of two nodes $e = (u, v)$. p_v is the active power *supply* ($p_v > 0$) or *demand* ($p_v < 0$) at node $v \in V$ (for a *neutral node* $p_v = 0$). We assume *pure reactive* lines, implying that each edge $e = (u, v) \in E$ is characterized by its *reactance* $x_e = x_{uv} = x_{vu} > 0$.

Given the power supply/demand vector $\vec{p} \in \mathbb{R}^{n\times 1}$ and the reactance values, a *power flow* is a solution $\vec{f} \in \mathbb{R}^{m\times 1}$ and $\vec{\theta} \in \mathbb{R}^{n\times 1}$ of:

$$\mathbf{A}\vec{\theta} = \vec{p}, \tag{1}$$

$$\mathbf{Y}\mathbf{D}^t\vec{\theta} = \vec{f}, \tag{2}$$

where $\mathbf{A} \in \mathbb{R}^{|V|\times|V|}$ is the *admittance matrix* of G,[1] defined as:

$$a_{uv} = \begin{cases} 0 & \text{if } u \neq v \text{ and } \{u,v\} \notin E, \\ -1/x_{uv} & \text{if } u \neq v \text{ and } \{u,v\} \in E, \\ -\sum_{w\in N(u)} a_{uw} & \text{if } u = v, \end{cases}$$

$\mathbf{D} \in \{-1, 0, 1\}^{n\times m}$ is the *incidence matrix* of G defined as,

$$d_{ij} = \begin{cases} 0 & \text{if } e_j \text{ is not incident to node } i, \\ 1 & \text{if } e_j \text{ is coming out of node } i, \\ -1 & \text{if } e_j \text{ is going into node } i, \end{cases}$$

and $\mathbf{Y} := \mathrm{diag}([1/x_{e_1}, 1/x_{e_2}, \ldots, 1/x_{e_m}])$ is a diagonal matrix with diagonal entries equal to the inverse of the reactance values. It is easy to see that $\mathbf{A} = \mathbf{D}\mathbf{Y}\mathbf{D}^t$.

Since $\mathbf{A}$ is not a full-rank matrix, we follow [10] and use the *pseudo-inverse* of $\mathbf{A}$, denoted by $\mathbf{A}^+$ to solve (1) as: $\vec{\theta} = \mathbf{A}^+\vec{p}$. Once $\vec{\theta}$ is computed, $\vec{f}$, can be obtained from (2).

Notation. Throughout this paper we use bold uppercase characters to denote matrices (e.g., $\mathbf{A}$), italic uppercase characters to denote sets (e.g., V), and italic lowercase characters and overline arrow to denote column vectors (e.g., $\vec{\theta}$). For a matrix $\mathbf{Q}$, q_{ij} denotes its $(i,j)^{\text{th}}$ entry, $\mathbf{Q}^t$ its transpose, and $\mathrm{tr}(\mathbf{Q})$ its trace. We denote the submatrix of $\mathbf{Q}$ limited to the first k columns by $\mathbf{Q}_k$ and the submatrix of $\mathbf{Q}$ limited to the first k rows and columns by $\mathbf{Q}_{k|k}$. For a column vector $\vec{y}$, $\vec{y}^t$ denotes its transpose, and $\vec{y}_k$ denotes the subvector of $\vec{y}$ with its first k entries. We use $\bar{k}$ to show the indices other that 1 to k (e.g., $\vec{f}_{\bar{k}}$ denotes the subvector of $\vec{f}$ with its $k+1$ to m entries).

[1] When $r_{uv} = 1\ \forall\{u,v\} \in E$, the admittance matrix $\mathbf{A}$ is the *Laplacian matrix* of the graph.

B. Failure Model

In this paper, we consider failures in a subset of size k of lines denoted by $L \subseteq E$. Without loss of generality, for convenience we assume $L = \{e_1, e_2, \ldots, e_k\}$. We denote the graph after failures by $G' = (V', E')$, in which $E' = E - L$ and $V' = V$. We also assume that removing edges in L from G does not disconnect the graph. Hence, G' is connected. Upon failures, the power flows redistributed in G' based on the equation $\mathbf{A}'\vec{\theta}' = \vec{p}$, in which $\mathbf{A}'$ is the admittance matrix of G'. Moreover, we define $\Delta\vec{f}_{\bar{k}} = \vec{f}_{\bar{k}} - \vec{f}'_{\bar{k}}$ to show the flow changes on the lines in $E\backslash L$ after the failure in lines in L.

It is easy to see that $\mathbf{A}' = \mathbf{A} - \mathbf{D}_k\mathbf{Y}_{k|k}\mathbf{D}_k^t$. In Section III, we use this equation to compute $\mathbf{A}'^+$ and quantify the effect of k-line failures.

C. Matrix of Equivalent Reactance Values

Define matrix $\mathbf{R} \in \mathbb{R}^{m\times m}$ as $\mathbf{R} := \mathbf{D}^t\mathbf{A}^+\mathbf{D}$. It is easy to see that for any $\forall 1 \leq i \leq m : r_{ii}$ is *equivalent reactance* between end buses of the line e_i. Matrix $\mathbf{R}$ is a symmetric matrix and very useful in quantifying the effect of line failures. In fact, in [10] we used this matrix to quantify the effect of single line failure when all the reactance values are equal to 1. In Section III, we generalize the idea in [10] for k-line failures.

III. Failure Analysis

In this section, we study the effect of k-line failures on the flow changes on the other lines. First, in the following lemma, we generalize the results in [10] for single line failures and provide an analytical update of the pseudo-inverse of the admittance matrix following a k-line failure.

Lemma 1: If G' is connected,

$$\mathbf{A}'^+ = \mathbf{A}^+ + \mathbf{A}^+\mathbf{D}_k\mathbf{Y}_{k|k}^{1/2}[\mathbf{I} - \mathbf{Y}_{k|k}^{1/2}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k\mathbf{Y}_{k|k}^{1/2}]^{-1}\mathbf{Y}_{k|k}^{1/2}\mathbf{D}_k^t\mathbf{A}^+.$$

Proof: First, from Lemma A.1 in the Appendix, since G' is connected, $[\mathbf{I} - \mathbf{Y}_{k|k}^{1/2}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k\mathbf{Y}_{k|k}^{1/2}]^{-1}$ is defined. Now to show the equality, it is easy to see that $\mathbf{A}\mathbf{A}^+ = \mathbf{I} - \frac{1}{n}\mathbf{J}$, in which $\mathbf{I}$ is the identity matrix and $\mathbf{J}$ is all 1 matrix (For more details see [10, Theorem 1]). Hence, from [16, Theorem 4.8], since $\mathbf{A}' = \mathbf{A} - \mathbf{D}_k\mathbf{Y}_{k|k}\mathbf{D}_k^t = \mathbf{A} - \mathbf{D}_k\mathbf{Y}_{k|k}^{1/2}(\mathbf{D}_k\mathbf{Y}_{k|k}^{1/2})^t$, the pseudo inverse of $\mathbf{A}'$ can be computed as,

$$\mathbf{A}'^+ = \mathbf{A}^+ + \mathbf{A}^+\mathbf{D}_k\mathbf{Y}_{k|k}^{1/2}[\mathbf{I} - \mathbf{Y}_{k|k}^{1/2}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k\mathbf{Y}_{k|k}^{1/2}]^{-1}\mathbf{Y}_{k|k}^{1/2}\mathbf{D}_k^t\mathbf{A}^+.$$

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From Lemma 1, the changes in phase angles after k-line failures can be computed as,

$$\begin{aligned}\vec{\theta}' - \vec{\theta} &= (\mathbf{A}'^{+} - \mathbf{A}^{+})\vec{p} \\ &= \mathbf{A}^{+}\mathbf{D}_k\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{D}^t_k\mathbf{A}^{+}\mathbf{D}_k\mathbf{Y}^{1/2}_{k|k}]^{-1}\mathbf{Y}^{1/2}_{k|k}\mathbf{D}^t_k\mathbf{A}^{+}\vec{p} \\ &= \mathbf{A}^{+}\mathbf{D}_k\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{D}^t_k\mathbf{A}^{+}\mathbf{D}_k\mathbf{Y}^{1/2}_{k|k}]^{-1}\mathbf{Y}^{-1/2}_{k|k}\vec{f}_k \\ &= \mathbf{A}^{+}\mathbf{D}_k\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\mathbf{Y}^{1/2}_{k|k}]^{-1}\mathbf{Y}^{-1/2}_{k|k}\vec{f}_k. \qquad (3)\end{aligned}$$

Using (3), we can compute the changes in the flows as,

$$\begin{aligned}\Delta\vec{f}_{\bar{k}} &= \mathbf{Y}_{\bar{k}|\bar{k}}\mathbf{D}^t_{\bar{k}}\mathbf{A}^{+}\mathbf{D}_k\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\mathbf{Y}^{1/2}_{k|k}]^{-1}\mathbf{Y}^{-1/2}_{k|k}\vec{f}_k \\ &= \mathbf{Y}_{\bar{k}|\bar{k}}\mathbf{R}_{\bar{k}|k}\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\mathbf{Y}^{1/2}_{k|k}]^{-1}\mathbf{Y}^{-1/2}_{k|k}\vec{f}_k. \qquad (4)\end{aligned}$$

It is important to see that $\mathbf{Y}_{\bar{k}|\bar{k}}\mathbf{R}_{\bar{k}|k}\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\mathbf{Y}^{1/2}_{k|k}]^{-1}\mathbf{Y}^{-1/2}_{k|k}$ is independent of $\vec{p}$ and solely depends on the structure properties of the network. Hence, following a similar definition in [9] for single line failures, we define this matrix as *k-line outage distribution matrix* and denote it by $\boldsymbol{\mathcal{L}} := \mathbf{Y}_{\bar{k}|\bar{k}}\mathbf{R}_{\bar{k}|k}\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\mathbf{Y}^{1/2}_{k|k}]^{-1}\mathbf{Y}^{-1/2}_{k|k}$. Hence, $\Delta\vec{f}_{\bar{k}} = \boldsymbol{\mathcal{L}}\vec{f}_k$.

While k-line outage distribution matrix captures all effects of a k-line failure on the flow changes, it is not efficient to compute and store it for contingency analysis in large power grids. In order to overcome this problem, we use the matrix of equivalent reactance values to efficiently compute the sum of changes in the power flows after k-line failures and to provide a metric to capture the essence of the flow changes after failures. The following lemma is the main step towards this goal. It demonstrates that $\mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2}$ is an idempotent matrix. We use this property, to provide the results in Corollaries 1 and 2.

Lemma 2: $\mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2} = \mathbf{Y}^{-1/2}\mathbf{D}^{+}\mathbf{D}\mathbf{Y}^{1/2}$, and therefore $(\mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2})^2 = \mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2}$.

Proof: We know from before that $\mathbf{R} = \mathbf{D}^t\mathbf{A}^{+}\mathbf{D}$ and $\mathbf{A} = \mathbf{D}\mathbf{Y}\mathbf{D}^t = (\mathbf{D}\mathbf{Y}^{1/2})(\mathbf{D}\mathbf{Y}^{1/2})^t$. Hence,

$$\begin{aligned}\mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2} &= \mathbf{Y}^{1/2}\mathbf{D}^t\mathbf{A}^{+}\mathbf{D}\mathbf{Y}^{1/2} = (\mathbf{D}\mathbf{Y}^{1/2})^t\mathbf{A}^{+}(\mathbf{D}\mathbf{Y}^{1/2}) \\ &= (\mathbf{D}\mathbf{Y}^{1/2})^t\big((\mathbf{D}\mathbf{Y}^{1/2})(\mathbf{D}\mathbf{Y}^{1/2})^t\big)^{+}(\mathbf{D}\mathbf{Y}^{1/2}) \\ &= (\mathbf{D}\mathbf{Y}^{1/2})^t\big((\mathbf{D}\mathbf{Y}^{1/2})^t\big)^{+}(\mathbf{D}\mathbf{Y}^{1/2})^{+}(\mathbf{D}\mathbf{Y}^{1/2}) \\ &= \big((\mathbf{D}\mathbf{Y}^{1/2})^{+}(\mathbf{D}\mathbf{Y}^{1/2})\big)^t(\mathbf{D}\mathbf{Y}^{1/2})^{+}(\mathbf{D}\mathbf{Y}^{1/2}).\end{aligned}$$

From the properties of the pseudo-inverse, $(\mathbf{D}\mathbf{Y}^{1/2})^{+}(\mathbf{D}\mathbf{Y}^{1/2})$ is a symmetric matrix. Moreover, $(\mathbf{D}\mathbf{Y}^{1/2})^{+}(\mathbf{D}\mathbf{Y}^{1/2})(\mathbf{D}\mathbf{Y}^{1/2})^{+} = (\mathbf{D}\mathbf{Y}^{1/2})^{+}$. Therefore,

$$\begin{aligned}\mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2} &= (\mathbf{D}\mathbf{Y}^{1/2})^{+}(\mathbf{D}\mathbf{Y}^{1/2})(\mathbf{D}\mathbf{Y}^{1/2})^{+}(\mathbf{D}\mathbf{Y}^{1/2}) \\ &= (\mathbf{D}\mathbf{Y}^{1/2})^{+}(\mathbf{D}\mathbf{Y}^{1/2}) = \mathbf{Y}^{-1/2}\mathbf{D}^{+}\mathbf{D}\mathbf{Y}^{1/2}.\end{aligned}$$

From this,

$$\begin{aligned}(\mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2})^2 &= (\mathbf{Y}^{-1/2}\mathbf{D}^{+}\mathbf{D}\mathbf{Y}^{1/2})^2 \\ &= \mathbf{Y}^{-1/2}\mathbf{D}^{+}\mathbf{D}\mathbf{D}^{+}\mathbf{D}\mathbf{Y}^{1/2} \\ &= \mathbf{Y}^{-1/2}\mathbf{D}^{+}\mathbf{D}\mathbf{Y}^{1/2} = \mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2}.\end{aligned}$$

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Corollary 1: $\mathbf{R}_{k|\bar{k}}\Delta\vec{f}_{\bar{k}} = \mathbf{R}_{k|k}\vec{f}_k$.

Proof: To make equations cleaner in the proof, define $\mathbf{H} := \mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2}$. From Lemma 2, $\mathbf{H}^2 = \mathbf{H}$. Hence, if we use block multiplication, then $\mathbf{H}_{k|k} = \mathbf{H}^2_{k|k} + \mathbf{H}_{k|\bar{k}}\mathbf{H}_{\bar{k}|k}$. Using this equation,

$$\begin{aligned}&\Delta\vec{f}_{\bar{k}} = \mathbf{Y}_{\bar{k}|\bar{k}}\mathbf{R}_{\bar{k}|k}\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\mathbf{Y}^{1/2}_{k|k}]^{-1}\mathbf{Y}^{-1/2}_{k|k}\vec{f}_k \\ \Rightarrow &\mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|\bar{k}}\Delta\vec{f}_{\bar{k}} = \mathbf{H}_{k|\bar{k}}\mathbf{H}_{\bar{k}|k}[\mathbf{I} - \mathbf{H}_{k|k}]^{-1}\mathbf{Y}^{-1/2}_{k|k}\vec{f}_k \\ \Rightarrow &\mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|\bar{k}}\Delta\vec{f}_{\bar{k}} = \mathbf{H}_{k|k}\mathbf{Y}^{-1/2}_{k|k}\vec{f}_k \\ \Rightarrow &\mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|\bar{k}}\Delta\vec{f}_{\bar{k}} = \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\vec{f}_k \Rightarrow \mathbf{R}_{k|\bar{k}}\Delta\vec{f}_{\bar{k}} = \mathbf{R}_{k|k}\vec{f}_k.\end{aligned}$$

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Corollary 1 shows the use of matrix $\mathbf{R}$ in evaluating the effect of k-line failures without computing the flows directly. This equation can be used to estimate the effect of k-line failures and is useful in reducing the total number of cases that are needed to be analyzed in contingency analysis. Since the matrix of equivalent reactance values needs to be computed only once, the matrix equation in Corollary 1 can be written for any k-line failures without further computations.

Although Corollary 1 does not depend on the computation of the flows after the failure, it still needs $O(m)$ operations to provide the vector of the flow changes. Hence, to quantify the effect of k-line failures more efficiently, in the following, we define a metric that captures the effect of k-line failures by a single value and show that it can be computed in $O(1)$.

We define $\Delta\vec{f}^t_{\bar{k}}\mathbf{Y}^{-1}_{\bar{k}|\bar{k}}\Delta\vec{f}_{\bar{k}}$ as the *disturbance value* of a failure. It is easy to see that $y^{-1}_{ii}\Delta f_i$ captures the changes in the phase angle differences between the end buses of a single line. Hence, the disturbance value $\Delta\vec{f}^t_{\bar{k}}\mathbf{Y}^{-1}_{\bar{k}|\bar{k}}\Delta\vec{f}_{\bar{k}} = \sum_{i=k+1}^{m} y^{-1}_{ii}\Delta f_i^2$ reflects both the big phase difference changes (which is important for the stability of the system) and the big flow changes (which is important for thermal safety of a line). In the following lemma, we provide the key step in computing the disturbance value of a failure analytically and efficiently in Corollary 2.

For convenience in equations, define $\mathbf{B} := [\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\mathbf{Y}^{1/2}_{k|k}]^{-1}$ and $\boldsymbol{\Phi} := \mathbf{Y}^{1/2}_{\bar{k}|\bar{k}}\mathbf{R}_{\bar{k}|k}\mathbf{Y}^{1/2}_{k|k}[\mathbf{I} - \mathbf{Y}^{1/2}_{k|k}\mathbf{R}_{k|k}\mathbf{Y}^{1/2}_{k|k}]^{-1}$. From (4), we know $\mathbf{Y}^{-1/2}_{\bar{k}|\bar{k}}\Delta\vec{f}_{\bar{k}} = \boldsymbol{\Phi}\mathbf{Y}^{-1/2}_{k|k}\vec{f}_k$.

Lemma 3: $\boldsymbol{\Phi}^t\boldsymbol{\Phi} = -\mathbf{I} + \mathbf{B}$.

Proof: To make equations cleaner in the proof, define $\mathbf{H} := \mathbf{Y}^{1/2}\mathbf{R}\mathbf{Y}^{1/2}$. From Lemma 2, $\mathbf{H}^2 = \mathbf{H}$. Hence, if we use block multiplication, then $\mathbf{H}_{k|k} = \mathbf{H}^2_{k|k} + \mathbf{H}_{k|\bar{k}}\mathbf{H}_{\bar{k}|k}$. Thus,

$$\begin{aligned}\mathbf{H}_{k|k} = \mathbf{H}^2_{k|k} + \mathbf{H}_{k|\bar{k}}\mathbf{H}_{\bar{k}|k} &\Rightarrow \mathbf{H}_{k|k}[\mathbf{I} - \mathbf{H}_{k|k}] = \mathbf{H}_{k|\bar{k}}\mathbf{H}_{\bar{k}|k} \\ &\Rightarrow \mathbf{H}_{k|k} = \mathbf{H}_{k|\bar{k}}\mathbf{H}_{\bar{k}|k}[\mathbf{I} - \mathbf{H}_{k|k}]^{-1}.\end{aligned}$$

It is easy to see that $\mathbf{\Phi} = \mathbf{H}_{\bar{k}|k}[\mathbf{I} - \mathbf{H}_{k|k}]^{-1}$. Hence, using equation above,

$$\begin{aligned}
\mathbf{\Phi}^t\mathbf{\Phi} &= [\mathbf{I} - \mathbf{H}_{k|k}]^{-1}\mathbf{H}_{k|\bar{k}}\mathbf{H}_{\bar{k}|k}[\mathbf{I} - \mathbf{H}_{k|k}]^{-1} \\
&= [\mathbf{I} - \mathbf{H}_{k|k}]^{-1}\mathbf{H}_{k|k} \\
&= [\mathbf{I} - \mathbf{H}_{k|k}]^{-1}\mathbf{H}_{k|k} - [\mathbf{I} - \mathbf{H}_{k|k}]^{-1} + [\mathbf{I} - \mathbf{H}_{k|k}]^{-1} \\
&= -[\mathbf{I} - \mathbf{H}_{k|k}]^{-1}[-\mathbf{H}_{k|k} + \mathbf{I}] + [\mathbf{I} - \mathbf{H}_{k|k}]^{-1} \\
&= -\mathbf{I} + [\mathbf{I} - \mathbf{H}_{k|k}]^{-1} = -\mathbf{I} + \mathbf{B}.
\end{aligned}$$

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Corollary 2: $\Delta\vec{f}_{\bar{k}}^{t}\mathbf{Y}_{\bar{k}|\bar{k}}^{-1}\Delta\vec{f}_{\bar{k}} = \vec{f}_k^t\mathbf{Y}_{k|k}^{-1/2}[-\mathbf{I}+\mathbf{B}]\mathbf{Y}_{k|k}^{-1/2}\vec{f}_k$.

Corollary 2 provides a very important tool for contingency analysis in the power grids. It shows that the disturbance value of a k-line failure can be computed in $O(k^3)$ time which is independent of the size of the network and only depends on the size of the initial failures. Notice that when $k << m$, $O(k^3) \approx O(1)$. Corollary 2 can be used for fast ranking of the contingencies based on the disturbance values and pruning most of the cases based on this value. This can significantly reduce the time complexity of the contingency analysis for large k. In the next section, we compute the disturbance values for all possible choices of 3-line failures in IEEE 118-bus and 300-bus systems and show that the disturbance values provide a very clear separation between the important cases and less important cases.

IV. Numerical Results

We computed the disturbance values for all possible choices of 3-line failures in IEEE 118-bus and 300-bus systems. Since, IEEE 118-bus system has 186 lines, it is easy to see that there are $\binom{186}{3} = 1055240$ possible choices for the initial set of failures. Using the method provided in Corollary 2, we could compute the disturbance values for all set of failures (and detect cases that disconnect the grid) in less than a minute. Out of those, 159591 of them make the graph disconnected. The cumulative distribution function of the disturbance values for the rest of 895649 cases are shown in Fig. 1. As can be seen, most of the cases do not result in a high (more than 10000) disturbance value. Only 4% of the cases (38130 of the cases) have a significant disturbance value. The figure suggests that our metric can provide a very clear separation between the failures with higher impact and lower impact.

Fig. 2 shows the cumulative distribution function of the disturbance values for all possible 3-line failures in IEEE 300-bus system that do not disconnect the grid (5473725 cases). As can be see, again most of the cases do not result in a high disturbance value and only less than 4% of the total cases have a significant disturbance value.[2]

The numerical results support our previous statement that the disturbance values can significantly reduce the total number of cases that are needed to be considered in the contingency analysis of the grids. Moreover, since Corollary 2 provides a very fast way of computing the disturbance values, our method can significantly decrease the time complexity of the contingency analysis in power grids.

[2]Notice that the threshold level for "high disturbance value" depends on the level of safety that a system operator likes to maintain. As can be seen in Figs. 1 and 2, more than 80% of the cases result in a very small disturbance value. Hence, selecting 1% to 20% of the cases with the highest disturbance value for deeper analysis is enough depending on the level of safety.

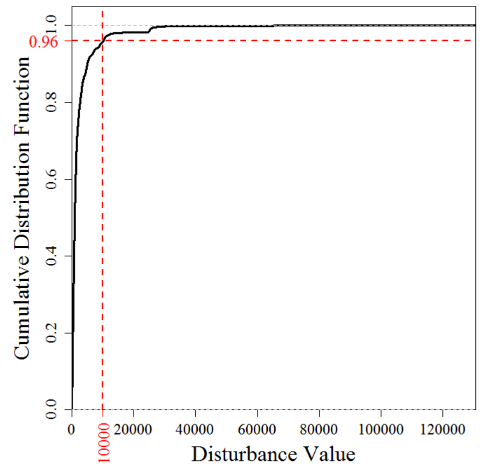

Fig. 1: The cumulative distribution function of the disturbance values for all 3-line failures in IEEE 118-bus system that does not disconnect the grid.

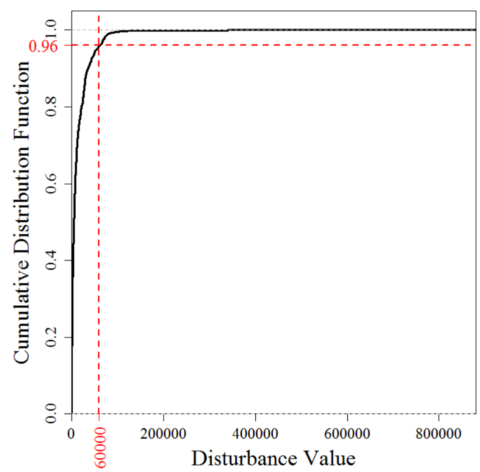

Fig. 2: The cumulative distribution function of the disturbance values for all 3-line failures in IEEE 300-bus system that does not disconnect the grid.

V. Conclusion

The results in this paper provide efficient tools for quantifying the effect of k-line failures. The most unique aspect of our approach is the use of the matrix of equivalent reactance values to efficiently capture the effect of k-line failures. We defined the disturbance value of a failure and show that this metric can be computed for any set of failures in $O(1)$. Our numerical results showed that disturbance values provide a clear separation between the failures with higher impact and lower impact. The tools we developed can be used to reduce the total number of cases needed to be analyzed in contingency analysis and can significantly reduce the computational complexity associated with this analysis.

While our method detects failures that make the grid disconnected, we plan to extend the definition of the disturbance value to these failures for ranking their severity. Moreover, although our results are comprehensive and support the DC power flow, they can be equally significant for the AC power flow model. Our goal is to extend the work presented here to flow changes after k-line failures for the AC power flow model.

Acknowledgement

This work was supported in part by CIAN NSF ERC under grant EEC-0812072, the People Programme (Marie Curie Actions) of the European Unions Seventh Framework Programme (FP7/2007-2013) under REA grant agreement no. [PIIF-GA-2013-629740].11, and DTRA grant HDTRA1-13-1-0021.

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Appendix A

The following lemma is the generalization of [10, Lemma 1] and similar to the idea used in [13, Theorem 2] to detect the connected components after multiple line failures. Our proof is different from the proof of a similar Theorem in [13, Theorem 2].

Lemma A.1: Matrix $\mathbf{I}-\mathbf{Y}_{k|k}^{1/2}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k\mathbf{Y}_{k|k}^{1/2}$ is invertible if, and only if G' is connected.

Proof: First, it is easy to see that $\mathbf{I}-\mathbf{Y}_{k|k}^{1/2}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k\mathbf{Y}_{k|k}^{1/2}$ is invertible if, and only if $\mathbf{I}-\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$ is invertible. Now assume, G' is disconnected. Without loss of generality, assume $C=\{e_1,e_2,\dots,e_r\}$ is a minimal subset of $\{e_1,e_2,\dots,e_k\}$ such that $G\backslash C$ is disconnected. Since, C is a minimal subset, $G\backslash C$ has only two connected components G_1 and G_2 and each $e_i\in C$ has one end in G_1 and the other end in G_2. Again with out loss of generality, assume that all the edges in C are directed from G_1 to G_2. We prove that vector $\vec{v}\in\{1,0\}^k$ defined as $v_i=y_{ii}$ for $i\le r$ and $v_i=0$ for $i>r$ is an eigenvector of $\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$ associated with the eigenvalue 1. Notice, that if $\vec{p}=\mathbf{D}_k\vec{v}$, then $\theta_i=1$ for $i\in G_1$ and $\theta_i=0$ for $i\in G_2$ gives a solution to DC power flow problem in G. It is easy to see that in this setting $\vec{f_k}=\vec{v}$. On the other hand, $\vec{f_k}=\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\vec{p}$, and since $\vec{f_k}=\vec{v}$ and $\vec{p}=\mathbf{D}_k\vec{v}$, therefore $\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k\vec{v}=\vec{v}$. Hence, $\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$ has eigenvalue 1 and $\mathbf{I}-\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$ is not invertible.

Now assume, $\mathbf{I}-\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$ is not invertible. Then, $\mathbf{I}-\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$ has an eigenvalue 0 and $\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$ has an eigenvalue 1. Assume $\vec{v}$ is the eigenvector associated with the eigenvalue 1 of $\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$. It is again easy to see that if $\vec{p}=\mathbf{D}_k\vec{v}$, then $\vec{f_k}=\vec{v}$ is the solution to the power flow problem in G. From the flow conservation equations, it is also easy to verify that $\vec{f_{\bar{k}}}=0$. Now, by contradiction assume G' is connected. Then, there should be a path in G' from a node i to node j such that $\theta_i\neq\theta_j$. Therefore, there should be an edge $e=(w,z)$ in this path such that $\theta_w\neq\theta_z$ and thus $f_e\neq 0$. However, since $e\in G'$ and $\vec{f_{\bar{k}}}=0$ we know that $f_e=0$ which is a contradiction. Therefore, G' is not connected.

From the proof it is easy to see that if $\vec{v}$ is an eigenvector associated with the eigenvalue 1 of $\mathbf{Y}_{k|k}\mathbf{D}_k^t\mathbf{A}^+\mathbf{D}_k$, then nodes with the same phase angle values in the solution of the power flow problem in G with $\vec{p}=\mathbf{D}_k\vec{v}$ form a connected component in G'. ∎