Doubly Balanced Connected Graph Partitioning

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We introduce and study the Doubly Balanced Connected graph Partitioning (DBCP) problem: Let $G{=}(V,E)$ be a connected graph with a weight (supply/demand) function $p{:}V\rightarrow\{-1,+1\}$ satisfying $p(V){=}\sum_{j\in V} p(j){=}0$. The objective is to partition G into (V_1,V_2) such that $G[V_1]$ and $G[V_2]$ are connected, $|p(V_1)|,|p(V_2)|\le c_p$, and $\max\{\frac{|V_1|}{|V_2|},\frac{|V_2|}{|V_1|}\}\le c_s$, for some constants c_p and c_s. When G is 2-connected, we show that a solution with $c_p{=}1$ and $c_s{=}2$ always exists and can be found in randomized polynomial time. Moreover, when G is 3-connected, we show that there is always a 'perfect' solution (a partition with $p(V_1){=}p(V_2){=}0$ and $|V_1|{=}|V_2|$, if $|V|{\equiv}0 (\bmod 4)$), and it can be found in randomized polynomial time. Our techniques can be extended, with similar results, to the case in which the weights are arbitrary (not necessarily ±1), and to the case that $p(V){\neq}0$ and the excess supply/demand should be split evenly. They also apply to the problem of partitioning a graph with two types of nodes into two large connected subgraphs that preserve approximately the proportion of the two types.

CCS Concepts: • **Mathematics of computing → Graph theory**; *Graph algorithms*;

Additional Key Words and Phrases: Graph Partitioning, Power Grid Islanding

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1 INTRODUCTION

Power Grid Islanding is an effective method to mitigate cascading failures in power grids [20]. The challenge is to partition the network into smaller connected components, called *islands*, such that each island can operate independently for a while. In order for an island to operate, it is necessary that the power supply and demand at that island are almost equal.[1] Equality of supply and demand in an island, however, may not be sufficient for its independent operation. It is also important that the infrastructure in that island has the physical capacity to safely transfer the power from the supply nodes to the demand nodes. When the island is large enough compared to the initial network, it is more likely that it has enough capacity. This problem has been studied in the power systems community but almost all the algorithms provided in the literature are heuristic methods that have been shown to be effective only by simulations [8, 16, 18, 20].

[1]If the supply and demand are not exactly equal but still relatively close, load shedding/generation curtailing can be used in order for the island to operate.

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Motivated by this application, we formally introduce and study the Doubly Balanced Connected graph Partitioning (DBCP) problem: Let $G=(V,E)$ be a connected graph with a weight (supply/demand) function $p:V\rightarrow\mathbb{Z}$ satisfying $p(V)=\sum_{j\in V}p(j)=0$. The objective is to partition V into (V_1,V_2) such that $G[V_1]$ and $G[V_2]$ are connected, $|p(V_1)|,|p(V_2)|\leq c_p$, and $\max\{\frac{|V_1|}{|V_2|},\frac{|V_2|}{|V_1|}\}\leq c_s$, for some constants c_p and c_s. We also consider the case that $p(V)\neq 0$, in which the excess supply/demand should be split roughly evenly.

The problem calls for a partition into two connected subgraphs that simultaneously balances two objectives, (1) the supply/demand within each part, and (2) the sizes of the parts. The connected partitioning problem with only the size objective has been studied previously. In the most well-known result, Lováz and Gyori [9, 14] independently proved, using different methods, that every k-connected graph can be partitioned into k arbitrarily sized connected subgraphs. However, neither of the proofs is constructive, and there are no known polynomial-time algorithms to find such a partition for $k>3$. For $k=2$, a linear time algorithm is provided in [21] and for $k=3$ an $O(|V|^2)$ algorithm is provided in [23].[2] The complexity of the problem with the size objective and related optimization problems have been studied in [3, 5, 6] and there are various NP-hardness and inapproximability results. Note that the size of the cut is not of any relevance here (so the extensive literature on finding balanced partitions, not necessarily connected, that minimize the cut is not relevant.)

The objective of balancing the supply/demand alone, when all $p(i)$ are ±1, can also be seen as an extension for the objective of balancing the size (which corresponds to $p(i)=1$). Our bi-objective problem of balancing both supply/demand and size, can be seen also as an extension of the problem of finding a partition that balances the size for two types of nodes simultaneously: Suppose the nodes of a graph are partitioned into red and blue nodes. Find a partition of the graph into two large connected subgraphs that splits approximately evenly both the red and the blue nodes.

We now summarize our results and techniques. Since the power grids are designed to withstand a single failure ("$N-1$" standard) [1], and therefore 2-connected, our focus is mainly on the graphs that are at least 2-connected. We first, in Section 4, study the connected partitioning problem with only the supply/demand balancing objective, and show results that parallel the results for balancing size alone, using similar techniques: The problem is NP-hard in general. For 2-connected graphs and weights $p(i)=\pm1$, there is always a perfectly balanced partition and we can find it easily using an st-numbering. For 3-connected graphs and weights $p(i)=\pm1$, there is a perfectly balanced partition into three connected graphs, and we can find it using a nonseparating ear decomposition of 3-connected graphs [4] and similar ideas as in [23].

The problem is more challenging when we deal with both balancing objectives, supply/demand and size. This is the main focus and occupies the bulk of this paper. Our main results are existence results and algorithms for 2- and 3-connected graphs. It is easy to observe that we cannot achieve perfection in one objective ($c_p=0$ or $c_s=1$) without sacrificing completely the other objective. We show that allowing the supply/demand of the parts to be off balance by at most the weight of one node suffices to get a partition that is roughly balanced also with respect to size.

First, in Section 4.1, we study the case of 3-connected graphs since we use this later as the basis of handling 2-connected graphs. We show that if $\forall i,\ p(i)=\pm1$, there is a partition that is perfectly balanced with respect to both objectives, if $|V|\equiv 0 \pmod 4$ (otherwise the sizes are slightly off for parity reasons); for general p, the partition is perfect in both objectives up to the weight of a single node. Furthermore, the partition can be constructed in randomized polynomial time. Our approach uses the convex embedding characterization of k-connectivity studied by Linial,

[2]For $k=2$, a much simpler approach than the one in [21] is to use the st-numbering [12] for 2-connected graphs.

Lováz, and Wigderson [13]. We need to adapt it for our purposes so that the convex embedding also has certain desired geometric properties, and for this purpose we use the nonseparating ear decomposition of 3-connected graphs of [4] to obtain a suitable embedding.

Then, in Section 4.2, we analyze the case of 2-connected graphs. We reduce it to two subcases: either (1) there is a separation pair that splits the graph into components that are not very large, or (2) we can perform a series of contractions to achieve a 3-connected graph whose edges represent contracted subgraphs that are not too large. We provide a good partitioning algorithm for case (1), and for case (2) we extend the algorithms for 3-connected graphs to handle also the complications arising from edges representing contracted subgraphs. Finally, in Section 5, we briefly discuss the problem of finding a connected partitioning of a graph with two types of nodes that splits roughly evenly both types.

2 DEFINITIONS AND BACKGROUND

In this section, we provide a short overview of the definitions and tools used in our work. Most of the graph theoretical terms used in this paper are relatively standard and borrowed from [2] and [24]. All the graphs in this paper are loopless.

2.1 Terms from Graph Theory

Cutpoints and Subgraphs: A *cutpoint* of a connected graph G is a node whose deletion results in a disconnected graph. Let X and Y be subsets of the nodes of a graph G. $G[X]$ denotes the subgraph of G *induced* by X. We denote by $E[X,Y]$ the set of edges of G with one end in X and the other end in Y. The neighborhood of a node v is denoted $N(v)$.

Connectivity: The connectivity of a graph $G{=}(V,E)$ is the minimum size of a set $S{\subset}V$ such that $G[V\backslash S]$ is not connected. A graph is k-connected if its connectivity is at least k.

2.2 st-numbering of a Graph

Given any edge $\{s,t\}$ in a 2-connected graph G, an *st-numbering* for G is a numbering for the nodes in G defined as follows [12]: the nodes of G are numbered from 1 to n so that s receives number 1, node t receives number n, and every node except s and t is adjacent both to a lower-numbered and to a higher-numbered node. It is shown in [7] that such a numbering can be found in $O(|V|{+}|E|)$.

2.3 Series-Parallel Graphs

A graph G is *series-parallel*, with terminals s and t, if it can be produced by a sequence of the following operations:

(1) Create a new graph, consisting of a single edge between s and t.
(2) Given two series parallel graphs, X and Y with terminals s_X, t_X and s_Y, t_Y respectively, form a new graph $G{=}P(X,Y)$ by identifying $s{=}s_X{=}s_Y$ and $t{=}t_X{=}t_Y$. This is known as the *parallel composition* of X and Y.
(3) Given two series parallel graphs X and Y, with terminals s_X, t_X and s_Y, t_Y respectively, form a new graph $G{=}S(X,Y)$ by identifying $s{=}s_X$, $t_X{=}s_Y$ and $t{=}t_Y$. This is known as the *series composition* of X and Y.

It is easy to see that a series-parallel graph is 2-connected if, and only if, the last operation is a parallel composition.

2.4 Nonseparating Induced Cycles and Ear Decomposition

Let H be a subgraph of a graph G. An *ear* of H in G is a nontrivial path in G whose ends lie in H but whose internal vertices do not. An ear decomposition of G is a decomposition $G{=}P_0{\cup}\cdots{\cup}P_k$ of the edges of G such that P_0 is a cycle and P_i for $i{\geq}1$ is an ear of $P_0{\cup}P_1{\cup}\cdots{\cup}P_{i-1}$ in G. It is known that every 2-connected graph has an ear decomposition (and vice-versa), and such a decomposition can be found in linear time.

A cycle C is a *nonseparating induced cycle* of G if $G\backslash C$ is connected and C has no chords. We say a cycle C avoids a node u, if $u{\notin}C$.

Theorem 2.1 (Tutte [22]). *Given a 3-connected graph $G(V,E)$ let $\{t,r\}$ be any edge of G and let u be any node of G, $r{\neq}u{\neq}t$. Then there is a nonseparating induced cycle of G through $\{t,r\}$ and avoiding u.*

Notice that since G is 3-connected in the previous theorem, every node in C has a neighbor in $G\backslash C$. Cheriyan and Maheshwari showed that the cycle in Theorem 2.1 can be found in $O(E)$ [4]. Moreover, they showed that any 3-connected graph G has a nonseparating ear decomposition $G{=}P_0{\cup}\cdots{\cup}P_k$ defined as follows: Let $V_i{=}V(P_0){\cup}V(P_1)\cdots{\cup}V(P_i)$, let $G_i{=}G[V_i]$ and $\overline{G}_i{=}G[V\backslash V_i]$. We say that $G{=}P_0{\cup}P_1{\cup}\cdots{\cup}P_k$ is an ear decomposition through edge $\{t,r\}$ and avoiding vertex u if the cycle P_0 contains edge $\{t,r\}$, and the last ear of length greater than one, say P_m, has u as its only internal vertex. An ear decomposition $P_0{\cup}P_1\cdots{\cup}P_k$ of graph G through edge $\{t,r\}$ and avoiding vertex u is a *nonseparating ear decomposition* if for all i, $0{\leq}i{<}m$, graph $\overline{G}_i$ is connected and each internal vertex of ear P_i has a neighbor in $\overline{G}_i$.

Theorem 2.2 (Cheriyan and Maheshwari [4]). *Given an edge $\{t,r\}$ and a vertex u of a 3-connected graph G, a nonseparating induced cycle of G through $\{t,r\}$ and avoiding u, and a nonseparating ear decomposition can be found in time $O(|V|{+}|E|)$.*

2.5 Partitioning of Graphs to Connected Subgraphs

The following theorem is the main existing result in partitioning of graphs into connected subgraphs and is proved independently by Lováz and Gyori [9, 14] by different methods.

Theorem 2.3 (Lováz and Gyori [9, 14]). *Let $G{=}(V,E)$ be a k-connected graph. Let $n{=}|V|$, $v_1,v_2,\ldots,v_k{\in}V$ and let $n_1,n_2,\ldots,n_k$ be positive integers satisfying $n_1{+}n_2{+}\cdots{+}n_k{=}n$. Then, there exists a partition of V into $(V_1,V_2\ldots,V_k)$ satisfying $v_i{\in}V_i$, $|V_i|{=}n_i$, and $G[V_i]$ is connected for $i{=}1,2,\ldots,k$.*

Although the existence of such a partition has long been proved, there is no polynomial-time algorithm to find such a partition for $k{>}3$. For $k{=}2$, it is easy to find such partition using st-numbering. For $k{=}3$, Wada and Kawaguchi [23] provided an $O(n^2)$ algorithm using the nonseparating ear decomposition of 3-connected graph.

2.6 Convex Embedding of Graphs

In this subsection, we provide a short overview of the beautiful work by Linial, Lováz, and Wigderson [13] on convex embedding of the k-connected graphs. Let $Q{=}\{q_1,q_2,\ldots,q_m\}$ be a finite set of points in $\mathbb{R}^d$. The convex hull conv(Q) of Q is the set of all points $\sum_{i=1}^{m}\lambda_i q_i$ with $\sum_{i=1}\lambda_i{=}1$. The rank of Q is defined by rank(Q)=1+dim(conv(Q)). Q is in general position if rank(S)=d+1 for every $(d{+}1)$-subset $S{\subseteq}Q$. Let G be a graph and $X{\subset}V$. A convex X-embedding of G is any mapping $f{:}V{\to}\mathbb{R}^{|X|-1}$ such that for each $v{\in}V\backslash X$, $f(v){\in}\text{conv}(f(N(v)))$. We say that the convex embedding is in general position if the set $f(V)$ of the points is in general position.

Theorem 2.4 (Linial, Lováz, and Wigderson [13]). *Let G be a graph on n vertices and $1{<}k{<}n$. Then the following two conditions are equivalent:*

1. *G is k-connected*
2. *For every $X{\subset}V$ with $|X|{=}k$, G has a convex X-embedding in general position.*

Notice that the special case of the Theorem for k=2 asserts the existence of an st-numbering of a 2-connected graph. The proof of this theorem is inspired by physics. The embedding is found by letting the edges of the graph behave like ideal springs and letting its vertices settle. A formal summary of the proof (1⟶2) is as follows (for more details see [13]). For each $v_i \in X$, define $f(v_i)$ arbitrary in $\mathbb{R}^{k-1}$ such that $f(X)$ is in general position. Assign to every edge $(u,v) \in E$ a positive elasticity coefficient c_{uv} and let $c \in \mathbb{R}^{|E|}$ be the vector of coefficients. It is proved in [13] that for almost any coefficient vector c, an embedding f that minimizes the potential function $P = \sum_{\{u,v\} \in E} c_{uv} \|f(u) - f(v)\|^2$ provides a convex X-embedding in general position ($\|.\|$ is the Euclidean norm). Moreover, the embedding that minimizes P can be computed as follows,

$$f(v) = \frac{1}{c_v} \sum_{u \in N(v)} c_{uv} f(u) \text{ for all } v \in V \setminus X,$$

in which $c_v = \sum_{u \in N(v)} c_{uv}$. Hence, the embedding can be found by solving a set of linear equations in at most $O(|V|^3)$ time (or matrix multiplication time).

3 BALANCING THE SUPPLY/DEMAND ONLY

In this section, we study the single objective problem of finding a partition of the graph into connected subgraphs that balances (approximately) the supply and demand in each part of the partition, without any regard to the sizes of the parts. We can state the optimization problem as follows, and will refer to it as the Balanced Connected Partitioning with Integer weights (BCPI) problem.

Definition 3.1. Given a graph $G=(V,E)$ with a weight (supply/demand) function $p : V \to \mathbb{Z}$ satisfying $\sum_{j \in V} p(j) = 0$. The BCPI problem is the problem of partitioning V into (V_1, V_2) such that

1. $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$,
2. $G[V_1]$ and $G[V_2]$ are connected,
3. $|p(V_1)| + |p(V_2)|$ is minimized, where $p(V_i) = \sum_{j \in V_i} p(j)$.

Clearly, the minimum possible value for $|p(V_1)| + |p(V_2)|$ that we can hope for is 0, which occurs iff $p(V_1) = p(V_2) = 0$. It is easy to show that the problem of determining whether there exists such a 'perfect' partition (and hence the BCPI problem) is strongly NP-hard. The proof is very similar to analogous results concerning the partition of a graph into two connected subgraphs with equal sizes (or weights, when nodes have positive weights) [3, 6]

Proposition 3.2. *(1) It is strongly NP-hard to determine whether there is a solution to the BCPI problem with value 0, even when G is 2-connected.*
(2) If G is not 2-connected, then this problem is NP-hard even when $\forall i, p(i) = \pm 1$.

Proof. We use the proof of [3, Theorem 2] with a modest change. The reduction is from the X3C problem [17], which is a variant of the *Exact Cover by 3-sets* and defined as follows: Given a set X with $|X|=3q$ and a family C of 3-element subsets of X such that $|C|=3q$ and each element of X appears in exactly 3 sets of C, decide whether C contains an exact cover for X. Given an instance (X,C) of $X3C$, let $G=(V,E)$ be the graph with the vertex set $V = X \cup C \cup \{a,b\}$ and edge set $E = \bigcup_{j=1}^{3q} [\{C_j x_i | x_i \in C_j\} \cup \{C_j a\} \cup \{C_j b\}]$. Set $p(a)=2q$, $p(b)=9q^2+q$, $p(C_j)=-1$, and $p(x_i)=-3q$. It is easy to verify that C contains an exact cover for X if and only if the BCPI problem has a solution such that $p(V_1)=p(V_2)=0$. On the one hand, if there is an exact cover $C' \subset C$, then we can let $V_1 = \{a\} \cup C \setminus C'$ and $V_2 = \{b\} \cup C' \cup X$; clearly both parts are connected, since C' covers X, and $p(V_1)=p(V_2)=0$ since $|C'|=q$. On the other if there is a connected partition (V_1, V_2) with $p(V_1)=p(V_2)=0$, then clearly a and b must be in different parts, say $a \in V_1$, $b \in V_2$. Part V_1 cannot contain any node in X, because then $p(V_1)$ would be negative. Hence V_1 consists of a and a subset C" of C with size $|C"|{=}2q$. Part V_2 must consist of the remaining nodes: b, $C'{=}C\backslash C"$ and X. Since $G[V_2]$ is connected, C' covers X, and since $|C'|{=}|C\backslash C"|{=}q$, C' is an exact cover of X. This shows the first claim.

For the second claim, attach to nodes a, b, and the x_is, paths of length $2q$, $9q^2{+}q$, and $3q$, respectively, and set the supply/demand values of a, b, the x_i's and the new nodes equal to $+1$ (for the paths for a and b) or -1 (for the x_i's). ■

Although it is NP-hard to tell whether there is a solution satisfying $p(V_1){=}p(V_2){=}0$, even when $\forall i, p(i){=}\pm 1$, in this case, if the graph G is 2-connected there is always such a solution. For general weights p, there is a solution such that $|p(V_1)|, |p(V_2)| \le \max_{j\in V} |p(j)|/2$ and it can be found easily in linear time using the st-numbering between two nodes.

Proposition 3.3. *Let G be a 2-connected graph and u, v any two nodes in V such that $p(u)p(v){>}0$.*
(1) There is a solution such that $u{\in}V_1$, $v{\in}V_2$, and $|p(V_1)|{=}|p(V_2)| \le \max_{j\in V} |p(j)|/2$.
(2) If $\forall i, p(i){=}\pm 1$, we can find a solution such that $u{\in}V_1$, $v{\in}V_2$, and $p(V_1){=}p(V_2){=}0$.
In both cases, the solution can be found in $O(|E|)$ time.

Proof. Clearly, part (2) follows immediately from part (1) because in this case, $p(V_1), p(V_2)$ are integer and $\max_{j\in V} |p(j)|/2{=}1/2$. To show part (1), pick two arbitrary nodes $u, v{\in}V$ with $p(u)p(v){>}0$. Since we want to separate u and v, we can assume wlog that initially $\{u, v\}{\in}G$. Since G is 2-connected, an st-numbering between nodes u and v as $u{=}v_1, v_2, \dots, v_n{=}v$ can be found in $O(|V|{+}|E|)$ [7]. Define $V_1^{(i)}{:=}\{v_1, v_2, \dots, v_i\}$. It is easy to see that $p(V_1^{(1)}){=}p(u){>}0$ and $p(V_1^{(n-1)}){=}-p(v){<}0$. Hence, there must exist an index $1{\le}i^*{<}n$ such that $|p(V_1^{(i^*)})|{>}0$ and $|p(V_1^{(i^*+1)})|{\le}0$. Since $|p(V_1^{(i)}){-}p(V_1^{(i+1)})|{=}|p(i^*{+}1)|$, either $|p(V_1^{(i^*)})|{\le}|p(i^*{+}1)|/2$ or $|p(V_1^{(i^*+1)})|{\le}|p(i^*{+}1)|/2$; Accordingly set $V_1{=}V_1^{(i^*)}$ or $V_1{=}V_1^{(i^*+1)}$. Let $V_2{=}V\backslash V_1$. Hence, (V_1, V_2) is a solution with $|p(V_1)|{=}|p(V_2)|{\le}|p(i^*{+}1)/2| {\le} \max_{j\in V} |p(j)|/2$. It is easy to see that i^* can be found in $O(|V|)$. ■

Remark 3.4. *The bound in Proposition 3.3 (1) is tight. A simple example is a cycle of length 4 like (v_1, v_2, v_3, v_4) with $p(v_1){=}{-}p$, $p(v_2){=}{-}p/2$, $p(v_3){=}p$, and $p(v_4){=}p/2$. It is easy to see that in this example $|p(V_1)|{+}|p(V_2)|{=}\max_{j\in V} |p(j)|{=}p$ is the best that one can do.*

3.1 Connected Partitioning into Many Parts

The BCPI problem can be extended to partitioning a graph into $k{=}3$ or more parts. Let $G{=}(V, E)$ be a graph with a weight function $p : V \longrightarrow \mathbb{Z}$ satisfying $\sum_{j\in V} p(j){=}0$. The BCPI$_k$ problem is the problem of partitioning G into $(V_1, V_2, \dots, V_k)$ such that for any $1{\le}i{\le}k$, $G[V_i]$ is connected and $\sum_{i=1}^{k} |p(V_i)|$ is minimized.

In the following proposition, we show that for $k{=}3$, if G is 3-connected and $p(i){=}\pm 1, \forall i$, then there is always a perfect partition (i.e., with $p(V_1){=}p(V_2){=}p(V_3){=}0$) and it can be found efficiently. For general p, we can find a partition such that $|p(V_1)|{+}|p(V_2)|{+}|p(V_3)| {\le} 2 \max_{j\in V} |p(j)|$. The proof and the algorithm use a similar approach as the algorithm in [23] for partitioning a 3-connected graph to three connected parts with prescribed sizes, using the nonseparating ear decomposition of 3-connected graphs as described in Subsection 2.4.

Proposition 3.5. *Let G be a 3-connected graph and u, v, w three nodes in V such that $p(u), p(v), p(w){>}0$ or $p(u), p(v), p(w){<}0$.*
(1) There is a solution such that $u{\in}V_1$, $v{\in}V_2$, $w{\in}V_3$, and $|p(V_1)|{+}|p(V_2)|{+}|p(V_3)| {\le} 2 \max_{j\in V} |p(j)|$.
(2) If $\forall i, p(i){=}\pm 1$, then there is a solution such that $u{\in}V_1$, $v{\in}V_2$, $w{\in}V_3$, and $|p(V_1)|{=}|p(V_2)|{=}|p(V_3)|{=}0$.
In both cases, the solution can be found in $O(|E|)$ time.

Proof. Consider the case of general function p, and let $p_{\max}=\max_{j\in V}|p(j)|$. We will show that we can find a solution such that $u\in V_1, v\in V_2, w\in V_3$ with $|p(V_1)|, |p(V_2)|\leq p_{\max}/2$. Since $|p(V_3)|=|p(V_1)+p(V_2)|$ (recall $p(V)=0$), this implies that $|p(V_3)|\leq p_{\max}$, and hence $|p(V_1)|+|p(V_2)|+|p(V_3)|\leq 2p_{\max}$. Furthermore, if $p(i)=\pm 1$ for all $i\in V$, hence $p_{\max}=1$, then $|p(V_1)|, |p(V_2)|\leq p_{\max}/2$ implies that $p(V_1)=p(V_2)=0$, and therefore also $p(V_3)=0$. Thus, both claims will follow.

Assume $u, v, w\in V$ and $p(u), p(v), p(w)>0$ (the case of negative $p(u), p(v), p(w)$ is symmetric). Since we want to separate u from v, we can assume without loss of generality that $\{u, v\}\in E$. Using Theorem 2.2, there is a non-separating ear decomposition through the edge $\{u, v\}$ and avoiding node w. Ignore the ears that do not contain any internal nodes, and let $Q_0\cup Q_1\cup\cdots\cup Q_r$ be the decomposition consisting of the ears with nodes; we have $w\in Q_r$. Let $V_i=V(Q_0)\cup V(Q_1)\cdots\cup V(Q_i)$, let $G_i=G[V_i]$ and $\overline{G}_i=G[V\setminus V_i]$. We distinguish two cases, depending on whether $p(V_0)\leq 0$ or $p(V_0)>0$.

(i) If $p(V_0)\leq 0$, then consider an st-numbering between u and v in V_0, say $u=v_1, v_2, \ldots, v_s=v$. Define $V_0^{(i)}=\{v_1, v_2, \ldots, v_i\}$. Since $p(u), p(v)>0$ and $p(V_0)\leq 0$, there must exist indices $1\leq i^*\leq j^*<s$ such that $p(V_0^{(i^*)})>0$, $p(V_0^{(i^*+1)})\leq 0$ and $p(V_0\setminus V_0^{(j^*+1)})>0$, $p(V_0\setminus V_0^{(j^*)})\leq 0$.

  (a) If $i^*=j^*$, since $p(V_0^{(i^*)})+p(v_{i^*+1})+p(V_0\setminus V_0^{(i^*+1)})=p(V_0)<0$, we have $p(V_0^{(i^*)})+p(V_0\setminus V_0^{(i^*+1)})\leq |p(v_{i^*+1})|$. Now, one of the following three cases happens:
    - If $p(V_0^{(i^*)})\leq|p(v_{i^*+1})|/2$ and $p(V_0\setminus V_0^{(i^*+1)})\leq|p(v_{i^*+1})|/2$, then it is easy to see that $V_1=V_0^{(i^*)}$, $V_2=V_0\setminus V_0^{(i^*+1)}$, and $V_3=V\setminus(V_1\cup V_2)$ is a good partition.
    - If $p(V_0^{(i^*)})>|p(v_{i^*+1})|/2$ and $p(V_0\setminus V_0^{(i^*+1)})\leq|p(v_{i^*+1})|/2$, then $p(V_0^{(i^*)})+p(v_{i^*+1})=p(V_0^{(i^*+1)})\leq|p(v_{i^*+1})|/2$. Hence, $V_1=V_0^{(i^*+1)}$, $V_2=V_0\setminus V_0^{(i^*+1)}$, and $V_3=V\setminus V_0$ is a good partition.
    - If $p(V_0^{(i^*)})\leq|p(v_{i^*+1})|/2$ and $p(V_0\setminus V_0^{(i^*+1)})>|p(v_{i^*+1})|/2$, then $p(V_0\setminus V_0^{(i^*+1)})+p(v_{i^*+1})=p(V_0\setminus V_0^{(i^*)})\leq|p(v_{i^*+1})|/2$. Hence, $V_1=V_0^{(i^*)}$, $V_2=V_0\setminus V_0^{(i^*)}$, and $V_3=V\setminus V_0$ is a good partition.

  (b) If $i^*<j^*$, then either $p(V_0^{(i^*)})\leq|p(v_{i^*+1})|/2$ or $|p(V_0^{(i^*+1)})|\leq|p(v_{i^*+1})|/2$, accordingly set $V_1=V_0^{(i^*)}$ or $V_1=V_0^{(i^*+1)}$. Similarly, either $p(V_0\setminus V_0^{(j^*+1)})\leq|p(v_{j^*+1})|/2$ or $|p(V_0\setminus V_0^{(j^*)})|\leq|p(v_{j^*+1})|/2$, so accordingly set $V_2=V_0\setminus V_0^{(j^*+1)}$ or $V_2=V_0\setminus V_0^{(j^*)}$. Set $V_3=V\setminus(V_1\cup V_2)$. It is easy to check that (V_1, V_2, V_3) is a good partition.

(ii) If $p(V_0)>0$, then since $p(w)>0$ and therefore $p(V_{r-1})<0$, there must exist an index $0\leq j<r-1$ such that $p(V_j)>0$ and $p(V_{j+1})\leq 0$. Consider an st-numbering between u and v in $G[V_j]$ as $u=v_1, v_2, \ldots, v_s=v$ and define $V_j^{(i)}=\{v_1, v_2, \ldots, v_i\}$. The ear Q_{j+1} is a path of new nodes $q_1, q_2, \ldots, q_t$ attached to two (distinct) nodes v_x, v_y of $G[V_j]$ through edges $\{v_x, q_1\}, \{q_t, v_y\}\in E$; assume wlog that $1\leq x<y\leq s$. For simplicity, we will use below Q_{j+1} to denote also the set $\{q_1, q_2, \ldots, q_t\}$ of internal (new) nodes of the ear. Also define $Q_{j+1}^{(i)}=\{q_1, q_2, \ldots, q_i\}$ and $Q_{j+1}^{(0)}=\emptyset$. One of the following cases must happen:

  (a) Suppose there is an index $1\leq i^*<(y-1)$ such that $p(V_j^{(i^*)})>0$ and $p(V_j^{(i^*+1)})\leq 0$ or there is an index $x+1<i^*<s$ such that $p(V_j\setminus V_j^{(i^*-1)})>0$ and $p(V_j\setminus V_j^{(i^*)})\leq 0$. Let's assume there is an index $1\leq i^*<(y-1)$, such that $p(V_j^{(i^*)})>0$ and $p(V_j^{(i^*+1)})\leq 0$ (the other case is exactly similar). Then either $p(V_j^{(i^*)})\leq|p(v_{i^*+1})|/2$ or $|p(V_j^{(i^*+1)})|\leq|p(v_{i^*+1})|/2$, accordingly set either $V_1=V_j^{(i^*)}$ or $V_1=V_j^{(i^*+1)}$. Set $V_2'=V_j\setminus V_1$. One of the following cases happens:
    - If $V_1=V_j^{(i^*)}$ and $p(V_2')\leq 0$, then since $p(V_j^{(i^*+1)})\leq 0$, we have $p(V_j\setminus V_j^{(i^*+1)})>0$. Hence, $p(V_2'\setminus\{v_{i^*+1}\})>0$. So, it is either $|p(V_2')|\leq|p(v_{i^*+1})|/2$ or $p(V_2'\setminus\{v_{i^*+1}\})\leq|p(v_{i^*+1})|/2$. Now if $p(V_2'\setminus\{v_{i^*+1}\})\leq|p(v_{i^*+1})|/2$, since also $p(V_1)\leq|p(v_{i^*+1})|/2$, $p(V_j)\leq 0$ which

contradicts with the assumption. Therefore, $|p(V_2')|\leq|p(v_{i^*+1})|/2$. Set $V_2=V_2'$ and $V_3=V\backslash(V_1\cup V_2)$. It is easy to check that (V_1,V_2,V_3) is a good partition.

- If $V_1=V_j^{(i^*)}$ and $p(V_2')>0$, then since $p(V_j\cup Q_{j+1})<0$, there is an index $0<t^*\leq t$, such that $p(V_2'\cup(Q_{j+1}\backslash Q_{j+1}^{(t^*)}))>0$ and $p(V_2'\cup(Q_{j+1}\backslash Q_{j+1}^{(t^*-1)}))\leq 0$. Hence, either $p(V_2'\cup(Q_{j+1}\backslash Q_{j+1}^{(t^*)}))\leq|p(q_{t^*})|/2$ or $|p(V_2'\cup(Q_{j+1}\backslash Q_{j+1}^{(t^*-1)}))|\leq|p(q_{t^*})|/2$, accordingly set $V_2=V_2'\cup(Q_{j+1}\backslash Q_{j+1}^{(t^*)})$ or $V_2=V_2'\cup(Q_{j+1}\backslash Q_{j+1}^{(t^*-1)})$. Set $V_3=V\backslash(V_1\cup V_2)$. It is easy to see that (V_1,V_2,V_3) is a good partition.
- If $V_1=V_j^{(i^*+1)}$, then since $p(V_1)\leq 0$, we have $p(V_2')>0$. The rest is exactly like the previous case when $V_1=V_j^{(i^*)}$ and $p(V_2')>0$.

(b) Suppose that for every $1\leq i<y$, $p(V_j^{(i)})>0$ and for every $x<i<s$, $p(V_j\backslash V_j^{(i)})>0$. Set $V_1'=V_j^{(y-1)}$ and $V_2'=V_j\backslash V_1'$. Based on the assumption $p(V_1'),p(V_2')>0$. Since $p(V_{j+1})\leq 0$, there are indices $0\leq i^*\leq j^*<t$ such that $p(V_1'\cup Q_{j+1}^{(i^*)})>0$, $p(V_1'\cup Q_{j+1}^{(i^*+1)})\leq 0$ and $p(V_2'\cup(Q_{j+1}\backslash Q_{j+1}^{(j^*+1)}))>0$, $p(V_2'\cup(Q_{j+1}\backslash Q_{j+1}^{(j^*)}))\leq 0$. The rest of the proof is similar to case (i) when $p(V_0)\leq 0$.

■

4 BALANCING BOTH OBJECTIVES

In this section, we formally define and study the Doubly Balanced Connected graph Partitioning (DBCP) problem.

Definition 4.1. Given a graph $G=(V,E)$ with a weight (supply/demand) function $p:V\longrightarrow\mathbb{Z}$ satisfying $p(V)=\sum_{j\in V}p(j)=0$ and constants $c_p\geq 0$, $c_s\geq 1$. The *DBCP problem* is the problem of partitioning V into (V_1,V_2) such that

(1) $V_1\cap V_2=\emptyset$ and $V_1\cup V_2=V$,
(2) $G[V_1]$ and $G[V_2]$ are connected,
(3) $|p(V_1)|,|p(V_2)|\leq c_p$ and $\max\{\frac{|V_1|}{|V_2|},\frac{|V_2|}{|V_1|}\}\leq c_s$, where $p(V_i)=\sum_{j\in V_i}p(j)$.

Remark 4.2. *Our techniques apply also to the case that $p(V)\neq 0$. In this case, the requirement 3 on $p(V_1)$ and $p(V_2)$ is $|p(V_1)-p(V)/2|,|p(V_2)-p(V)/2|\leq c_p$, i.e., the excess supply/demand is split approximately evenly between the two parts.*

We will concentrate on 2-connected and 3-connected graphs and show that we can construct efficiently good partitions. For most of the section we will focus on the case that $p(i)=\pm 1,\forall i\in V$. This case contains all the essential ideas. All the techniques generalize to the case of arbitrary p, and we will state the corresponding theorems.

We observed in Section 2 that if the graph is 2-connected and $p(i)=\pm 1,\forall i\in V$ then there is always a connected partition that is perfect with respect to the weight objective, $p(V_1)=p(V_2)=0$, i.e., (3) is satisfied with $c_p=0$. We know also from [9, 14] that there is always a connected partition that is perfect with respect to the size objective, $|V_1|=|V_2|$, i.e., condition 3 is satisfied with $c_s=1$. The following observations show that combining the two objectives makes the problem more challenging. If we insist on $c_p=0$, then c_s cannot be bounded in general, (it will be $\Omega(|V|)$), and if we insist on $c_s=1$, then c_p cannot be bounded. The series-parallel graphs of Figure 1 provide simple counterexamples.

Observation 4.3. *If $c_p=0$, then for any $c_s<|V|/2-1$, there exist a 2-connected graph G such that the DBCP problem does not have a solution even when $\forall i,p(i)=\pm 1$.*

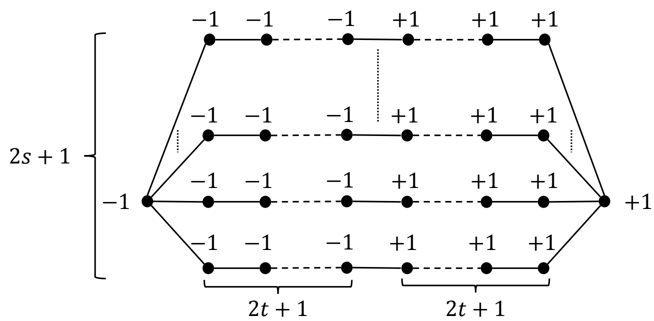


Fig. 1. Series-parallel graphs with $2s{+}1$ paths of length $4t{+}2$ used in Observations 4.3 and 4.4.

Proof. In the graph depicted in Figure 1, set $t{=}0$. Since the parts should be connected and we want $c_p{=}0$, the left -1 terminal node should be in the same part as the right $+1$ terminal node. It is easy to see that the only way to partition the graph into two connected parts such that the left -1 node and the right $+1$ node end up in the same part is to have the two nodes and $2s$ paths between them in one part, and a single path (consisting of $4t{+}2{=}2$ nodes) in another part. Hence, the only connected partition of this graph with $c_p{=}0$ results in $c_s{=}(4s{+}2)/2{=}2s{+}1{=}|V|/2{-}1$. ■

Observation 4.4. *If $c_s{=}1$, then for any $c_p{<}|V|/6$, there exist a 2-connected graph G such that the DBCP problem does not have a solution even when $\forall i, p(i){=}\pm 1$.*

Proof. In the graph depicted in Figure 1, set $s{=}1$; thus there are only 3 parallel paths. Since we want $c_s{=}1$, it is easy to see that the left -1 terminal node and the right $+1$ terminal node cannot be in the same part, otherwise, to have two connected parts, one part should include 2 paths and the two terminal nodes which results in $c_s{\neq}1$. Now, if we want the two terminal nodes to be in separate parts to have $c_s{=}1$, it is easy to verify that to minimize c_p, exactly one of the paths should be included in the part with the left terminal node and one of the paths should be included in the part with the right terminal node. Moreover, the last remaining path should be divided in half between the two parts. For such a partition $c_p{=}2t{+}2{>}|V|/6$. ■

Thus, c_p has to be at least 1 to have any hope for a bounded c_s. We show in this section that $c_p{=}1$ suffices for all 2-connected graphs. We first treat 3-connected graphs.

4.1 3-Connected Graphs

Let $G{=}(V, E)$ be a 3-connected graph. Assume for the most of this section that $\forall i, p(i){=}\pm 1$ and $p(V){=}0$ (we will state the results for general p at the end). We show that G has a partition that is essentially perfect with respect to both objectives, i.e., with $c_p{=}0$ and $c_s{=}1$. We say "essentially", because $p(V_1){=}p(V_2){=}0$ and $|V_1|{=}|V_2|$ imply that $|V_1|{=}|V_2|$ are even, and hence V must be a multiple of 4. If this is the case, then indeed we can find such a perfect partition. If $|V|{\equiv}2(\text{mod } 4)$ ($|V|$ has to be even since $p(V){=}0$), then we can find an 'almost perfect' partition, one in which $|p(V_1)|{=}|p(V_2)|{=}1$ and $|V_1|{=}|V_2|$ (or one in which $p(V_1){=}p(V_2){=}0$ and $|V_1|{=}|V_2|{+}2$).

We first treat the case that G contains a triangle (i.e., cycle of length 3). In the following Lemma, we use the embedding for k-connected graphs introduced in [13] and as described in Subsection 2.6, to show that if G is 3-connected with a triangle and all weights are ± 1, then the DBCP problem has a perfect solution.

Lemma 4.5. *If G is 3-connected with a triangle, $\forall i, p(i)=\pm 1$, and $|V|\equiv 0 (\mathrm{mod}\ 4)$, then there exists a solution to the DBCP problem with $p(V_1)=p(V_2)=0$ and $|V_1|=|V_2|$. If $|V|\equiv 2 (\mathrm{mod}\ 4)$, then there is a solution with $p(V_1)=p(V_2)=0$ and $|V_1|=|V_2|+2$. Moreover, this partition can be found in randomized polynomial time.*

Proof. Assume that $|V|\equiv 0 (\mathrm{mod}\ 4)$; the proof for the case $|V|\equiv 2 (\mathrm{mod}\ 4)$ is similar. Following Theorem 2.4, if G is a k-connected graph, then for every $X\subset V$ with $|X|=k$, G has a convex X-embedding in general position. Moreover, this embedding can be found by solving a set of linear equations of size $|V|$. Now, assume $v, u, w\in V$ form a triangle in G. Set $X=\{v, u, w\}$. Using the theorem, G has a convex X-embedding $f: V\rightarrow\mathbb{R}^2$ in general position. Consider a circle C around the triangle $f(u), f(v), f(w)$ in $\mathbb{R}^2$ as shown in an example in Fig. 2. Also consider a directed line $\mathcal{L}$ that is tangent to the circle C at point A and is not perpendicular to any line connecting any two of the nodes. If we project the nodes of G onto the line $\mathcal{L}$, since the embedding is convex and also $\{u, v\}, \{u, w\}, \{w, v\}\in E$, the order of the nodes' projection gives an st-numbering between the first and the last node (notice that the first and last nodes are always from the set X). In particular, from the definition of convex embedding, since each node in $V\backslash\{u, v, w\}$ is in the convex hull of its neighbors, when it is projected on line $\mathcal{L}$, at least one of its neighbors should be projected on its right and one of its neighbors should be projected on its left. Moreover, since $\{u, w, v\}$ are pairwise connected, whichever of the three nodes ends up in the middle of the other two in the projection, has a neighbor on its right and a neighbor on its left. Hence, the order of the nodes' projection gives an st-numbering between the first and the last node. For instance in Fig. 2, the order of projections give an st-numbering between the nodes u and v in G. Hence, if we set V_1 to be the $|V|/2$ nodes whose projections come first and V_2 are the $|V|/2$ nodes whose projections come last, then $G[V_1]$ and $G[V_2]$ are both connected and $|V_1|=|V_2|=|V|/2$. The only thing that may not match is $p(V_1)$ and $p(V_2)$.

Notice that for each directed line tangent to the circle C, we can similarly get a partition such that $|V_1|=|V_2|=|V|/2$. So all we need is a point D on the circle C such that if we partition based on the directed line tangent to C at point D, then $p(V_1)=p(V_2)=0$. To find such a point, we move $\mathcal{L}$ from being tangent at point A to point B, where AB is a diameter of the circle C, and consider the resulting partition. Notice that if at point A, $p(V_1)>0$, then at point B since V_1 and V_2 completely switch places compared to the partition at point A, $p(V_1)<0$. Hence, as we move $\mathcal{L}$ from being tangent at point A to point B and keep it tangent to the circle, in the resulting partitions, $p(V_1)$ goes from some positive value to a non-positive value. Notice that as we rotate the line $\mathcal{L}$ around the circle, the ordering of two nodes in the projection changes only when $\mathcal{L}$ becomes perpendicular to the line that connects the embeddings of the two nodes on the plane. The partition (V_1, V_2) changes only if the last node i of V_1 in the projected ordering switches places with the first node j of V_2 in the ordering, and this happens if $\mathcal{L}$ becomes perpendicular to the line that connects $f(i)$ to $f(j)$. Since the embedding is in general position, no other node is collinear with $f(i)$ and $f(j)$, hence no other node has the same projection on $\mathcal{L}$ as i and j. Therefore, V_1 changes at most by one node leaving V_1 and one node entering V_1 at each step as we move $\mathcal{L}$. Hence, $p(V_1)$ changes by either ± 2 or 0 value at each change. Now, since $|V|\equiv 0 (\mathrm{mod}\ 4)$, $p(V_1)$ has an even value in all the resulting partitions. Therefore, as we move $\mathcal{L}$ from being tangent at point A to point B, there must be a point D such that in the resulting partition $p(V_1)=p(V_2)=0$.

Note that the order of the projected points and V_1 may change only when a line that passes through 2 nodes of graph G is perpendicular to $\mathcal{L}$. We can sort first the slopes of all the lines connecting two nodes of G (or alternatively we can use a priority queue) and then rotate the line $\mathcal{L}$ from the initial position A until we find the point D that yields a partition with $p(V_1)=p(V_2)=0$ in $O(|V|^2 \log |V|)$ time. ■

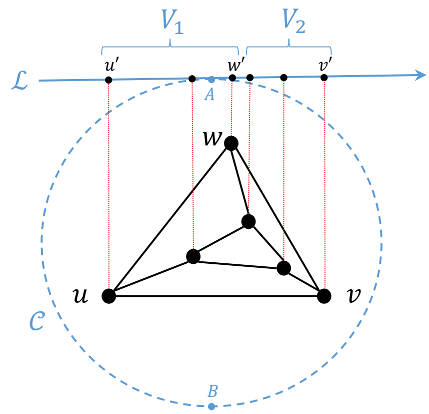


Fig. 2. Proof of Lemma 4.5.

When G is a triangle-free 3-connected graph, however, the proof in Lemma 4.5 cannot be directly used anymore. The reason is if for example $\{u, v\} \notin E$ and we project the nodes of G onto the line $\mathcal{L}$, this time the order of the nodes projection may not give necessarily an st-numbering between the first and the last node. For example, if u and w are the first and last nodes, then since v is not connected to u, then it does not necessarily have a neighbor on its left in the projection. To prove a similar result for triangle-free 3-connected case, we first provide the following two Lemmas. The main purpose of the following two Lemmas are to compensate for the triangle-freeness of G in the proof of Lemma 4.5. The idea is to show that in every 3-connected graph, there is a triple $\{u, w, v\} \in V$, such that $\{u, w\}, \{w, v\} \in E$ and in every partition that we get by the approach used in the proof of Lemma 4.5, if u and v are in V_i, so is a path between u and v.

Lemma 4.6. *If G is 3-connected, then there exists a set $\{u, v, w\} \in V$ and a partition of V into (V_1', V_2') such that:*

(1) $V_1' \cap V_2' = \emptyset$ *and* $V_1' \cup V_2' = V$,
(2) $G[V_1']$ *and* $G[V_2']$ *are connected,*
(3) $\{u, w\}, \{v, w\} \in E$,
(4) $w \in V_1'$, $u, v \in V_2'$, *and* u, v *are not cutpoints of* $G[V_2']$,
(5) $|V_2'| \leq |V|/2$.

Moreover, such a partition and $\{u, v, w\}$ can be found in $O(|E|)$ time.

Proof. Using the algorithm presented in [4], we can find a non-separating cycle C_0 in G such that every node in C_0 has a neighbor in $G \setminus C_0$ in $O(|E|)$ time. Now, we consider two cases:

(i) If $|C_0| \leq |V|/2+1$, then select any three consecutive nodes (u, w, v) of C_0 and set $V_2' = C_0 \setminus \{w\}$ and $V_1' = V \setminus V_2'$.
(ii) If $|C_0| > |V|/2+1$, since every node in C_0 has a neighbor in $G \setminus C_0$, there exists a node $w \in V \setminus C_0$ such that $|N(w) \cap C_0| \geq 2$. We can select such a node w, two adjacent nodes $u, v \in N(w) \cap C_0$, and the shorter path P in C_0 from u to v, such that $|P| \leq |V|/2$. Set $V_2' = P$ and $V_1' = V \setminus V_2'$. ∎

Lemma 4.7. *Given a partition (V_1', V_2') of a 3-connected graph G with following properties:*

(1) $V_1' \cap V_2' = \emptyset$ *and* $V_1' \cup V_2' = V$,
(2) $G[V_1']$ *and* $G[V_2']$ *are connected,*
(3) $w \in V_1'$, $u, v \in V_2'$, *and* u, v *are not cutpoints of* $G[V_2']$,

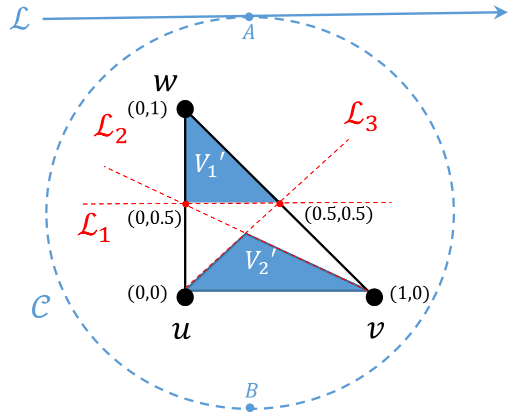


Fig. 3. Proof of Lemma 4.7 and Theorem 4.8.

G has a convex X-embedding in general position with mapping $f{:}V{\rightarrow}\mathbb{R}^2$ such that:

(1) *$X{=}\{u, w, v\}$, $f(u){=}(0, 0)$, $f(v){=}(1, 0)$, and $f(w){=}(0, 1)$,*
(2) *Every node i in V_1' is mapped to a point $(f_1(i), f_2(i))$ with $f_2(i){\geq}1/2$,*
(3) *Every node i in V_2' is mapped to a point $(f_1(i), f_2(i))$ with $f_1(i){\geq}f_2(i)$ and $f_1(i){+}2f_2(i){\leq}1$.*

Moreover, such an embedding can be found in randomized polynomial time.

Proof. Set $X{=}\{v, u, w\}$. Using [13], G has a convex X-embedding in $\mathbb{R}^2$ in general position with mapping $f{:}V{\rightarrow}\mathbb{R}^2$ such that $f(u){=}(0, 0)$, $f(v){=}(1, 0)$, and $f(w){=}(0, 1)$. In the X-embedding of the nodes, we have a freedom to set the elasticity coefficient vector $\vec{c}$ to anything that we want (except a measure zero set of vectors). So for any edge $\{i, j\}{\in}G[V_1']{\cup}G[V_2']$, set $c_{ij}{=}g$; and for any $\{i, j\}{\in}E[V_1', V_2']$, set $c_{ij}{=}1$. Assume $\mathcal{L}_1$ is the line $y{=}0.5$, $\mathcal{L}_2$ is the line $x{+}2y{=}1$, and $\mathcal{L}_3$ is the line $x{=}y$.

First, we show that there exist a g for which all the nodes in V_1' will be embedded above the line $\mathcal{L}_1$. To show this, from [13], we know the embedding is such that it minimizes the total potential $P(f, \vec{c}){=}\sum_{\{i,j\}\in E} c_{ij}\|f(i){-}f(j)\|^2$. Notice that we can independently minimize P on x-axis values and y-axis values as below:

$$\begin{aligned} \min_f P &= \min_{f_1} P_x + \min_{f_2} P_y \\ &= \min_{f_1} \sum_{\{i,j\}\in E} c_{ij}(f_1(i){-}f_1(j))^2 + \min_{f_2} \sum_{\{i,j\}\in E} c_{ij}(f_2(i){-}f_2(j))^2 \end{aligned}$$

Now, notice that if we place all the nodes in V_1' at point (0,1) and all the nodes in V_2' on the line uv, then $P_y{\leq}|E|$. Hence, if f_2 minimizes P_y, then $P_y(f_2, c){\leq}|E|$. Set $g{\geq}4|V|^2|E|$. We show that if f_2 minimizes P_y, then for all edges $\{i, j\}{\in}G[V_1']{\cup}G[V_2']$, $(f_2(i){-}f_2(j))^2{\leq}1/(4|V|^2)$. By contradiction, assume there is an edge $\{i, j\}{\in}G[V_1']{\cup}G[V_2']$ such that $(f_2(i){-}f_2(j))^2{>}1/(4|V|^2)$. Then, $c_{ij}(f_2(i){-}f_2(j))^2{=}g(f_2(i){-}f_2(j))^2{>}|E|$. Hence, $P_y(f_2, c){>}|E|$ which contradicts with the fact the f_2 minimizes P_y. Therefore, if $g{\geq}4|V|^2|E|$, then for all $\{i, j\}{\in}G[V_1']{\cup}G[V_2']$, $|f_2(i){-}f_2(j)|{\leq}1/(2|V|)$. Now, since $G[V_1']$ is connected, all the nodes in V_1' are connected to w with a path of length (in number of hops) less than $|V|{-}1$. Hence, using the triangle inequality, for all $i{\in}V_1'$:

$$|f_2(w){-}f_2(i)|{\leq}(|V|{-}1)/(2|V|){<}1/2{\Longrightarrow}|1{-}f_2(i)|{<}1/2,$$

which means that all the nodes in V_1' are above $\mathcal{L}_1$.

With the very same argument, if $g{\geq}t^2|V|^2|E|$, then for all $i{\in}V_2'$, $f_2(i){<}1/t$.

Now, we want to prove that there is a g such that all the nodes in V_2' will be embedded below the lines $\mathcal{L}_2$ and $\mathcal{L}_3$. Define $n_1(i){:=}|N(i){\cap}V_1'|$ and $n_2(i){:=}|N(i){\cap}V_2'|$. From [13], we know the embedding is such that for all $i{\in}V\backslash\{u,v,w\}$, $f(i){=}1/c_i\sum_{j\in N(i)}c_{ij}f(j)$, where $c_j{=}\sum_{j\in N(i)}c_{ij}f(j)$. Since $G[V_2']$ is connected and u is not a cutpoint of $G[V_2']$, for every $i{\in}V_2'\backslash\{u,v\}$ there is a path $i{=}v_1,v_2,\ldots,v_r{=}v$ in V_2' not containing node u. Using this ordering:

$$\begin{cases} f_1(v_j){\geq}\frac{1}{n_2(v_j)g+n_1(v_j)}gf_1(v_{j+1}){\geq}(1/|V|)f_1(v_{j+1}), & \forall j{\in}\{1,\ldots,r{-}1\} \\ f_1(v_r){=}f_1(v){=}1 \end{cases}$$
$$\Rightarrow \forall i{\in}V_2'\backslash\{u,v\}\colon f_1(i){\geq}(1/|V|)^r{\geq}(1/|V|)^{|V|}.$$

On the other hand, from the previous part, if we set $g{\geq}|V|^{2|V|+2}|E|$, then for all $i{\in}V_2'$, $f_2(i){\leq}(1/|V|)^{|V|}$. Hence, for all $i{\in}V_2'$, $f_2(i){\leq}f_1(i)$, which means that all the nodes in V_2' will be placed below the line $\mathcal{L}_3$.

With the very same idea, we show that there exist a g for which all the nodes in V_2' will be placed below the line $\mathcal{L}_2$. Since $G[V_2']$ is connected and v is not a cutpoint of $G[V_2']$, for every $i{\in}V_2'\backslash\{u,v\}$ there is a path $u{=}u_1,u_2,\ldots,u_t{=}i$ in V_2' not containing node v. Notice that for all $i{\in}V\backslash\{u,v,w\}$, $1{-}f_1(i){=}1/c_i\sum_{j\in N(i)}c_{ij}(1{-}f_1(j))$. Hence, since $\forall j{\in}V{:}f_1(j){\leq}1$, we have,

$$\begin{cases} 1{-}f_1(u_j){\geq}\frac{1}{n_2(u_j)g+n_1(u_j)}g(1{-}f_1(u_{i-1})){\geq}(1/|V|)(1{-}f_1(u_{i-1})), & \forall j{\in}\{2,\ldots,t\} \\ 1{-}f_1(u){=}1{-}f_1(u_1){=}1 \end{cases}$$
$$\Rightarrow \forall i{\in}V_2'\backslash\{u,v\}\colon 1{-}f_1(i){\geq}(1/|V|)^t{\geq}(1/|V|)^{|V|}.$$

From the previous part, if we set $g{\geq}4|V|^{2|V|+2}|E|$, then for all $i{\in}V_2'$, $f_2(i){\leq}1/2(1/|V|)^{|V|}$. Hence, for $i{\in}V_2'$, $f_1(i){+}2f_2(i){\leq}1$, which means that all the nodes in V_2' will be placed below the line $\mathcal{L}_3$. Therefore, if we set $g{\geq}4|V|^{2|V|+2}|E|$, then we will get an embedding as depicted in Fig. 3. Note that a polynomial number of bits suffices for g.

Notice that if $\vec{c}$ is a "good" vector, then so is $\vec{c}{+}\vec{\epsilon}$ in which $\vec{\epsilon}$ is a vector with very small Euclidean norm. Hence, we can always find a "good" vector $\vec{c}$ which results in a X-embedding in general position. ∎

Using Lemmas 4.6 and 4.7, we are now able to prove that for any 3-connected graph G such that all the weights are ± 1, the DBCP problem has a solution for $c_p{=}0$ and $c_s{=}1$. The idea of the proof is similar to the proof of Lemma 4.5, however, we need to use Lemma 4.6 to find a desirable partition (V_1',V_2') and then use this partition to find an embedding with properties as described in Lemma 4.7. By using this embedding, we can show that in every partition that we obtain by the approach in the proof of Lemma 4.5, if u and v are in V_i, so is a path between u and v. This implies then that $G[V_1]$ and $G[V_2]$ are connected. So we can use similar arguments as in the proof of Lemma 4.5 to prove the following theorem.

Theorem 4.8. *If G is 3-connected, $\forall i, p(i){=}\pm 1$, and $|V|{\equiv}0(\text{mod } 4)$, then there exists a solution to the DBCP problem with $p(V_1){=}p(V_2){=}0$ and $|V_1|{=}|V_2|$. If $|V|{\equiv}2(\text{mod } 4)$, then there is a solution with $p(V_1){=}p(V_2){=}0$ and $|V_1|{=}|V_2|{+}2$. Moreover, this partition can be found in randomized polynomial time.*

Proof. Assume that $|V|{\equiv}0(\text{mod } 4)$; the proof for the case $|V|{\equiv}2(\text{mod } 4)$ is similar. Using Lemma 4.6, we can find $\{u,v,w\}{\in}V$ and a partition (V_1',V_2') of V with properties described in the Lemma. Set $X{=}\{u,v,w\}$. Using Lemma 4.7, we can find a convex X-embedding of G in general position with properties described in the Lemma as depicted in Fig. 3. The rest of the proof is very similar to the proof of Lemma 4.5. We consider again a circle C around $f(u), f(v), f(w)$ in $\mathbb{R}^2$ as shown in Fig. 3.

Also consider a directed line $\mathcal{L}$ tangent to the circle C at point A. If we project the nodes of G onto the line $\mathcal{L}$, this time the order of the nodes projection gives an st-numbering between the first and the last node only if u and v are the first and last node. However, if we set V_1 to be the $|V|/2$ nodes whose projections come first and V_2 are the $|V|/2$ nodes whose projections come last, then $G[V_1]$ and $G[V_2]$ are both connected even when u and v are not the first and last nodes. The reason lies on the special embedding that we considered here. Assume for example w and v are the first and the last projected nodes, and V_1 and V_2 are set of the $|V|/2$ nodes which projections come first and last, respectively. Two cases might happen:

(i) If $u, w \in V_1$ and $v \in V_2$, then since $\{u, w\} \in E$, both $G[V_1]$ and $G[V_2]$ are connected because of the properties of the embedding.

(ii) If $w \in V_1$ and $u, v \in V_2$, since $|V_2'| \leq |V|/2$ and $|V_2| = |V|/2$, then either $V_2 = V_2'$ or $V_2 \cap V_1' \neq \emptyset$. If $V_2 = V_2'$, and hence $V_1 = V_1'$ then there is nothing to prove. So assume there is a node $z \in V_2 \cap V_1'$. From the properties of the embedding, the triangle $\{z, u, v\}$ contains all the nodes of V_2'. Since $\{z, u, v\} \in V_2$, and V_2 contains all the nodes that are on a same side of a halfplane, we should also have $V_2' \subset V_2$. Now, from the properties of the embedding, it is easy to see that every node in V_2 has a path either to u or v. Since $V_2' \subset V_2$, there is also a path between u and v. Thus, $G[V_2]$ is connected. From the properties of the embedding, $G[V_1]$ is connected as before.

The rest of the proof is exactly the same as the proof of Lemma 4.5. We move $\mathcal{L}$ from being tangent at point A to point B (AB is a diameter of the circle C) and consider the resulting partition. Notice that if at point A, $p(V_1) > 0$, then at point B since V_1 and V_2 completely switch places compared to the partition at point A, $p(V_1) < 0$. Hence, as we move $\mathcal{L}$ from being tangent at point A to point B and keep it tangent to the circle, in the resulted partitions, $p(V_1)$ goes from some positive value to a negative value. Notice that the partition (V_1, V_2) changes only if $\mathcal{L}$ passes a point D on the circle such that at D, $\mathcal{L}$ is perpendicular to a line that connects $f(i)$ to $f(j)$ for $i, j \in V$. Now, since the embedding is in general position, there are exactly two points on every line that connects two points $f(i)$ and $f(j)$, so V_1 changes at most by one node leaving V_1 and one node entering V_1. Hence, $p(V_1)$ changes by either ± 2 or 0 value at each change. Now, since $|V| \equiv 0 (\mathrm{mod}\ 4)$, $p(V_1)$ has an even value in all the resulting partitions. Therefore, as we move $\mathcal{L}$ from being tangent at point A to point B, there should be a point D such that in the resulted partition $p(V_1) = p(V_2) = 0$. ∎

It is easy to check for a 3-connected graph G, by using the same approach as in the proof of Lemma 4.5 and Theorem 4.8, that even when the weights are arbitrary (not necessarily ± 1) and also $p(V) \neq 0$, we can still find a solution (a connected partition) (V_1, V_2) for G with $c_p = \max_{i \in V} |p(i)|$ and $c_s = 1$.

Corollary 4.9. *If G is 3-connected, then the DBCP problem (with arbitrary p, and not necessarily satisfying $p(V) = 0$) has a solution (V_1, V_2) with $|p(V_1) - p(V)/2|, |p(V_2) - p(V)/2| \leq \max_{i \in V} |p(i)|$ and $|V_1| = |V_2|$ if $|V|$ is even ($|V_1| = |V_2| + 1$ if $|V|$ is odd). Moreover, this solution can be found in randomized polynomial time.*

Proof. The proof is similar to the above proof for the case of ± 1 weights. We give first the proof for even $|V|$. Fix a convex embedding of the graph satisfying the properties of Lemma 4.7. Consider a circle C as above, a tangent line $\mathcal{L}$ that is not perpendicular to any line connecting two nodes, and consider the corresponding partition (V_1, V_2) of the nodes into two equal halves. As we showed above, $G[V_1], G[V_2]$ are connected. Assume without of loss of generality that the partition for the tangent at the starting point A has $p(V_1) \geq p(V)/2$. If $p(V_1) = p(V)/2$ then we are done, so assume $p(V_1) > p(V)/2$, and hence $p(V_2) < p(V)/2$. At the antidiametric point B, the roles of V_1 and V_2 are switched, therefore at B we have $p(V_1)<p(V)/2$. As we move the tangent line from A to B, the value of $p(V_1)$ starts above $p(V)/2$ and ends below $p(V)/2$. In each step where V_1 changes, the change consists of replacing one node by another node, hence $p(V_1)$ changes at most by $2p_{\max}$, where $p_{\max}=\max_{i\in V}|p(i)|$. Look at the step in which $p(V_1)$ changes from a value greater than $p(V)/2$ to a value $\leq p(V)/2$. If before this step $p(V_1)>p(V)/2+p_{\max}$, then after this step $p(V_1)>p(V)/2+p_{\max}-2p_{\max}=p(V)/2-p_{\max}$. Thus, either before or after this step, the partition satisfies $|p(V_1)-p(V)/2|\leq p_{\max}$, and hence also $|p(V_2)-p(V)/2|\leq p_{\max}$.

The proof for odd $|V|$ is similar. For a tangent line $\mathcal{L}$ that is not perpendicular to any line connecting two nodes, we let V_1 be the set of the first $\lceil V/2\rceil$ nodes in the projected ordering and V_2 the remaining set of the last $\lfloor V/2\rfloor$ nodes. Let m be the middle node in the ordering, let $V_1'=V_1\setminus\{m\}$ and $V_2'=V_2\cup\{m\}$. As above, $G[V_1]$ and $G[V_2]$ are connected (and so are $G[V_1']$ and $G[V_2']$). Assume without loss of generality that the tangent at point A yields a partition with $p(V_1)\geq p(V)/2$. If $p(V_1)\leq p(V)/2+p_{\max}$ then we are done: (V_1,V_2) is a good partition. So assume $p(V_1)>p(V)/2+p_{\max}$. This implies that $p(V_1')>p/2$ and hence $p(V_2')<p(V)/2$. At the antidiametric point B, the role of V_1 (i.e. the set of the first $\lceil V/2\rceil$ nodes in the ordering) is played by the set V_2' of the partition at A, which has value smaller than $p(V)/2$. The rest of the argument now is the same: as we rotate the tangent from point A to point B, the value $p(V_1)$ starts greater than $p(V)/2+p_{\max}$, it ends smaller than $p(V)/2$, and it changes in each step at most by $2p_{\max}$. Therefore at some points it satisfies $|p(V_1)-p(V)/2|\leq p_{\max}$, and hence at that point also $|p(V_2)-p(V)/2|\leq p_{\max}$. ■

4.2 2-Connected Graphs

We first define a *pseudo-path* between two nodes in a graph as below. The definition is inspired by the definition of the st-numbering.

Definition 4.10. A *pseudo-path* between nodes u and v in $G=(V,E)$, is a sequence of nodes $v_1,\ldots,v_t$ such that if $v_0=u$ and $v_{t+1}=v$, then for any $1\leq i\leq t$, v_i has neighbors v_j and v_k such that $j<i<k$. Note that the pseudo-path does not include the ending points u and v. If $P=v_1,\ldots,v_t$ is a pseudo-path between nodes u and v, then $|P|=t$ denotes the number of nodes in the pseudo-path.

Using the pseudo-path notion, in the following lemma we show that if G is 2-connected and has a separation pair such that none of the resulting components are too large, then the DBCP problem always has a solution for some $c_p=c_s=O(1)$. The idea used in the proof of this lemma is one of the building blocks of the proof for the general 2-connected graph case.

Lemma 4.11. *Given a 2-connected graph G, if $\forall i{:}p(i)=\pm1$ and G has a separation pair $\{u,v\}\subset V$ such that for every connected component $H_i=(V_{H_i},E_{H_i})$ of $G[V\setminus\{u,v\}]$, $|V_{H_i}|<\lfloor 2|V|/3\rfloor$, then the DBCP problem has a solution for $c_p=1$, $c_s=2$, and it can be found in $O(|E|)$ time.*

Proof. There is a separation pair $\{u,v\}\in V$ such that if $H_1,\ldots,H_k$ are the connected components of $G\setminus\{u,v\}$, for every i, $|V_{H_i}|<\lfloor 2|V|/3\rfloor$. Since G is 2-connected, $H_1,\ldots,H_k$ can be represented by pseudo-paths $P_1,\ldots,P_k$ between u and v (just consider an st-numbering between u and v and use that numbering to form the pseudo-paths). Assume $P_1,\ldots,P_k$ are in increasing order based on their lengths. We can partition the pseudo-paths into two subsets S_1 and S_2 such that $S_1\cap S_2=\emptyset$, $S_1\cup S_2=\{P_1,\ldots,P_k\}$ and $\sum_{P_j\in S_i}|P_j|\geq\lceil|V|/3\rceil-1$ for $i=1,2$. The proof is very simple. Add greedily pseudo-paths in order to S_1 until $\sum_{P_j\in S_1}|P_j|$ becomes at least $\lceil|V|/3\rceil-1$. Let $S_1=\{P_1,\ldots,P_i\}$, and $S_2=\{P_{i+1},\ldots,P_k\}$. Since $|P_k|<\lfloor 2|V|/3\rfloor$, we have $i<k$ and $S_2\neq\emptyset$. We have to show that $\sum_{P_j\in S_2}|P_j|\geq\lceil|V|/3\rceil-1$. If $|P_k|\geq\lceil|V|/3\rceil-1$, then the claim really holds. If $|P_k|<\lceil|V|/3\rceil-1$, then $|P_i|\leq|P_k|<\lceil|V|/3\rceil-1$, and $|P_1|+\cdots+|P_{i-1}|<\lceil|V|/3\rceil-1$ implies that $|P_{i+1}|+\cdots+|P_k|\geq\lceil|V|/3\rceil-1$.

Now, if we put all the pseudo-paths in S_1 back to back, they will form a longer pseudo-path Q_1 between u and v. Similarly, we can form another pseudo-path Q_2 from the pseudo-paths in

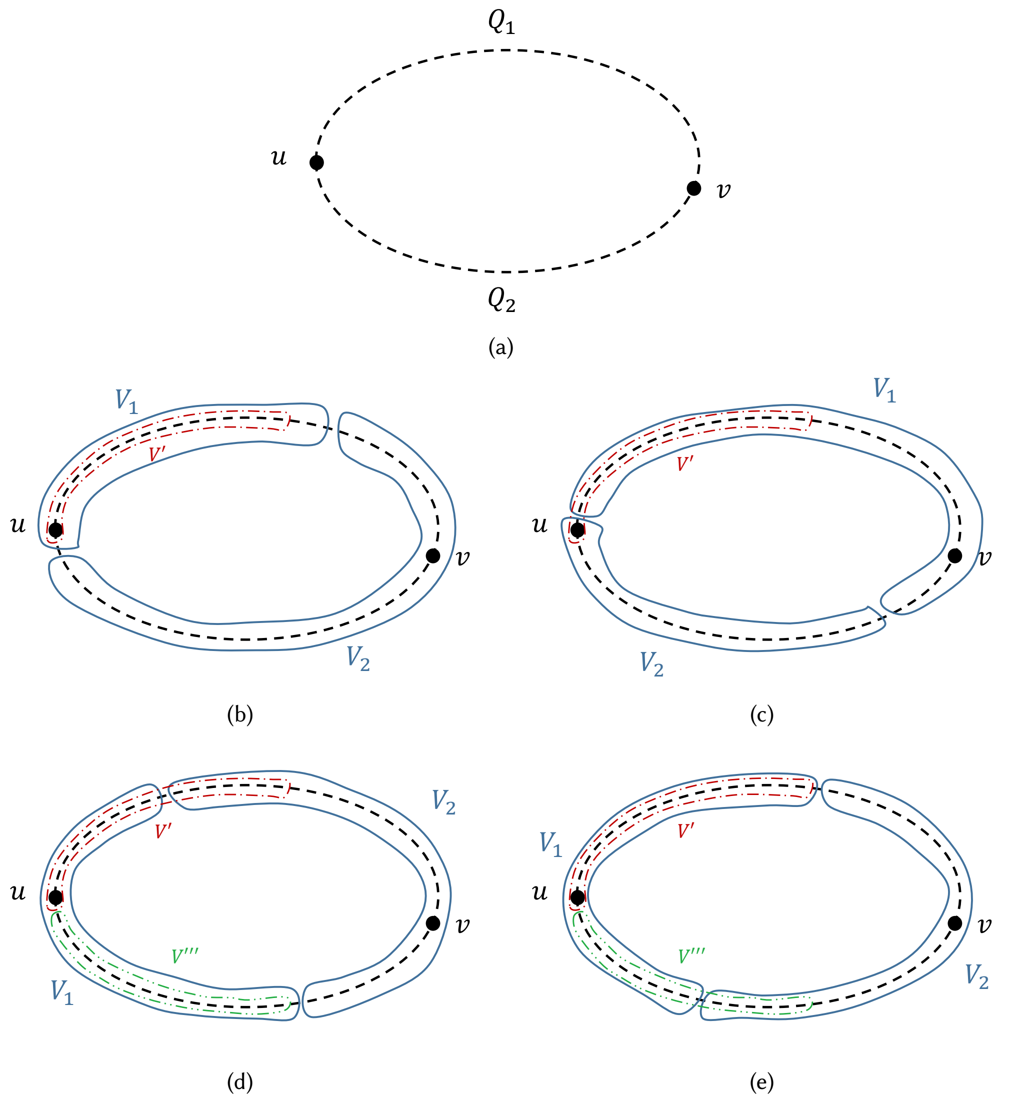


Fig. 4. Proof of Lemma 4.11.

S_2 (Fig. 4a). Without loss of generality we can assume $|Q_1| \geq |Q_2|$. From u, including u itself, we count $\lceil |V|/3 \rceil$ of the nodes in Q_1 towards v and put them in a set V'. Without loss of generality, we can assume $p(V') \geq 0$. If $p(V')=0$, then $(V', V \backslash V')$ is a good partition and we are done. Hence, assume $p(V')>0$. We keep V' fixed and make a new set V'' by adding nodes from Q_1 to V' one by one before we get to v. If $p(V'')$ hits 0 as we add nodes one by one, we stop and let $V_1 = V''$ and $V_2 = V \backslash V''$, then (V_1, V_2) is a good partition and we are done (Fig. 4b).

So, assume $V'' = Q_1 \cup \{u\}$ and $p(V'')>0$. Since $|Q_2 \cup \{v\}| \geq \lceil |V|/3 \rceil$, $|V''| \leq \lfloor 2|V|/3 \rfloor$. If $|V''| < \lceil 2|V|/3 \rceil$, we add nodes from $Q_2 \cup \{v\}$ one by one toward u until either $p(V'')=0$ or $|V''| = \lceil 2|V|/3 \rceil$. If we hit 0 first, or at the same time, i.e., $p(V'')=0$ and $|V''| \leq \lceil 2|V|/3 \rceil$), define $V_1 = V'' \backslash \{u\}$; then $(V_1, V \backslash V_1)$ is a good partition (Fig. 4c). So assume $|V''| = \lceil 2|V|/3 \rceil$ and $p(V'')>0$. Define $V''' = V \backslash V''$. Since $p(V'')>0$

and $|V''|=\lceil 2|V|/3\rceil$, then $p(V''')<0$ and $|V'''|=\lfloor |V|/3\rfloor$. Also notice that $V'''\subseteq Q_2$. We consider two cases. Either $|p(V')|\geq|p(V''')|$ or $|p(V')|<|p(V''')|$.

If $|p(V')|\geq|p(V''')|$ we start from u and pick nodes one by one from Q_1 in order until we get a subset $V_1'\neq\emptyset$ of V' such that $p(V_1')=|p(V''')|$ (i.e., $p(V_1'\cup V''')=0$). Define $V_1=V_1'\cup V'''$. Then $p(V_1)=0$, $|V_1|\geq|V'''|+1\geq\lceil |V|/3\rceil$ and $|V_1|\leq 2\lfloor |V|/3\rfloor$ (note that $|V_1|$ is even since $p(V_1)=0$ and $|V_1|\leq|V'\cup V'''|=\lfloor |V|/3\rfloor+\lceil |V|/3\rceil$). Hence, $(V_1, V\backslash V_1)$ is a good partition (Fig. 4d).

If $|p(V')|<|p(V''')|$, the argument is similar. We can build a new set V_1 by adding nodes one by one from V''' to V' until $p(V_1)=0$; then $|V_1|\leq\lfloor 2|V|/3\rfloor$. Hence, $(V_1, V\backslash V_1)$ is a good partition (Fig. 4e). ∎

Corollary 4.12. *If G is a 2-connected series-parallel graph and $\forall i{:}p(i)=\pm 1$, then the DBCP problem has a solution for $c_p=1$, $c_s=2$, and the solution can be found in $O(|E|)$ time.*

Proof. Every series-parallel graph G has a separation pair $\{u, v\}$ such that every connected component of $G[V\backslash\{u, v\}]$ has less than $\lfloor 2|V|/3\rfloor$ nodes, and furthermore, such a separation pair can be found in linear time. To see this, consider the derivation tree T of the construction of G. The root of T corresponds to G, the leaves correspond to the edges, and every internal node i corresponds to a subgraph $G_i=(V_i, E_i)$ that is the series or parallel composition of the subgraphs corresponding to its children. Starting at the root of T, walk down the tree following always the edge to the child corresponding to a subgraph with the maximum number of nodes until the number of nodes becomes $\leq\lceil 2|V|/3\rceil$. Thus, we arrive at a node i of the tree such that $|V_i|>\lceil 2|V|/3\rceil$ and $|V_j|\leq\lceil 2|V|/3\rceil$ for all children j of i. Let u_i, v_i be the terminals of G_i. Note that u_i, v_i separate all the nodes of G_i from all the nodes that are not in G_i. Since $|V_i|>\lceil 2|V|/3\rceil$, we have $|V\backslash V_i|<|V|/3$. If G_i is the parallel composition of the graphs corresponding to the children of i, then the separation pair $\{u_i, v_i\}$ has the desired property, i.e. all the components of $G[V\backslash\{u, v\}]$ have less than $\lfloor 2|V|/3\rfloor$ nodes.

Suppose G_i is the series composition of the graphs G_j, G_k corresponding to the children j, k of i, and let w be the common terminal of G_j, G_k; thus, G_i has terminals u_i, w, and G_k has terminals w, v_i. Assume wlog that $|V_j|\geq|V_k|$. Then $\lceil |V|/3\rceil<|V_j|\leq\lceil 2|V|/3\rceil$. The pair $\{u_i, w\}$ of terminals of G_j separates all the nodes of $V_j\backslash\{u_i, w\}$ from all the nodes of $V\backslash V_j$, and both these sets have less than $\lfloor 2|V|/3\rfloor$ nodes. Thus, $\{u_i, w\}$ has the required property. ∎

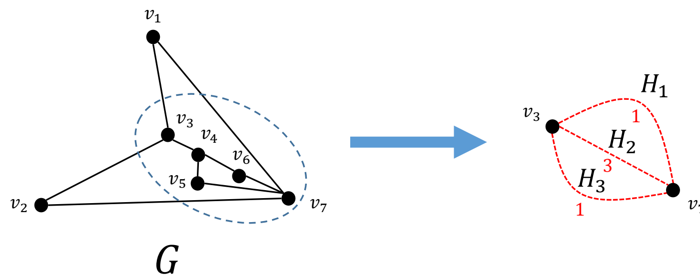
The graph in Figure 1 with $s=1$ shows that these parameters are the best possible for series parallel graphs: if $c_p=O(1)$ then c_s must be at least 2.

To generalize Lemma 4.11 to all 2-connected graphs, we need to define the *contractible* subgraph and the *contraction* of a given graph as below.

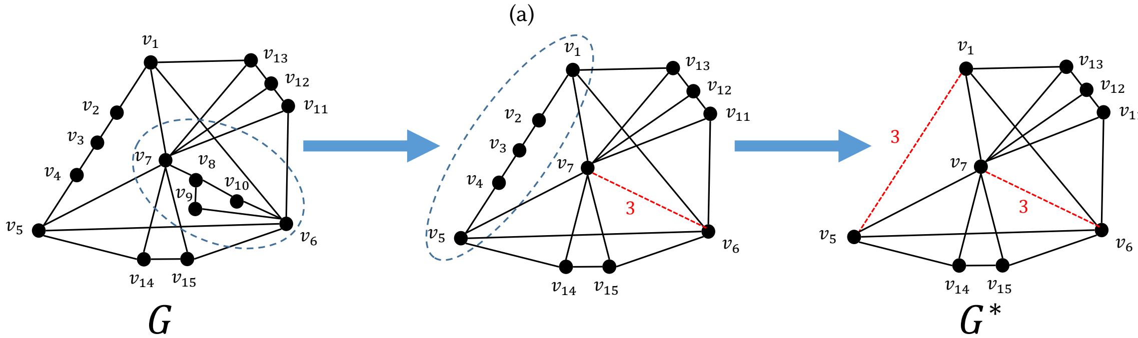
Definition 4.13. We say an induced subgraph H of a 2-connected graph G is *contractible*, if there is a separating pair $\{u, v\}\subset V$ such that $H=(V_H, E_H)$ is a connected component of $G[V\backslash\{u, v\}]$. Moreover, if we replace H by a weighted edge e' with weight $\mathrm{w}(e')=|V_H|$ between the nodes u and v in G to obtain a smaller graph G', we say G is *contracted* to G'.

Remark 4.14. *Notice that every contractible subgraph of a 2-connected graph G can also be represented by a pseudo-path between its associated separating pair. We use this property in the proof of Theorem 4.18.*

Using the notion of the graph contraction, in the following lemma, we show that to partition a 2-connected graph, we can reduce it to one of two cases: either G can be considered as a graph with a set of short pseudo-paths between two nodes, or it can be contracted into a 3-connected graph as illustrated in Fig. 5.



(a)



(b)

Fig. 5. Lemma 4.15.

Lemma 4.15. *In every 2-connected graph G=(V, E), one of the following cases holds, and we can determine which in $O(|E|)$ time:*

(1) *There is a separation pair $\{u, v\}{\subset}V$ such that if $H_1, \ldots, H_k$ are the connected components of $G[V\setminus\{u, v\}]$, for all i, $|V_{H_i}|{<}\lfloor 2|V|/3\rfloor$.*
(2) *After a set of contractions, G can be transformed into a 3-connected graph G^*=(V^*, E^*) with weighted edges representing contracted subgraphs such that for every $e^*{\in}E^*$, $\mathrm{w}(e^*){\leq}\lceil|V|/3\rceil{-}2$.*

Proof. If there is no separation pairs in G, then G is 3-connected and there is nothing left to prove. So assume $\{u, v\}{\subset}V$ is a separation pair and $H_1, \ldots, H_k$ are the connected components of $G[V\setminus\{u, v\}]$. If $\forall i$, $|V_{H_i}|{<}\lfloor 2|V|/3\rfloor$, we are done. So let's assume there is a connected component H_j such that $|V_{H_j}|{\geq}\lfloor 2|V|/3\rfloor$. Then all the other components H_i, $i{\neq}j$, can be contracted and represented by an edge of weight at most $\lceil|V|/3\rceil{-}2$ between u and v. Now, we repeat the process by considering the weight of the edges in the size of each connected component (a weighted edge can be contracted again as part of a new connected component and its weight will be added to the total number of nodes in that connected component). An example for each case is shown in Fig. 5 for q=3. We can find either a suitable separation pair as in case 1 or a suitable contracted graph G^* as in case 2 in linear time using the Hopcroft-Tarjan algorithm for finding the 3-connected components [10]. ■

Definition 4.16. In a graph G^* with weighted edges representing contracted subgraphs, define the weight for a subset of nodes $U^*{\subset}V^*$ as $\mathrm{w}(U^*)$=$\mathrm{w}(G^*[U^*])$:=$|U^*|$+$\sum_{e\in G^*[U^*]}\mathrm{w}(e)$.

The following lemma strengthens and extends Lemma 4.6 to weighted graphs.

Lemma 4.17. *If $G^*{=}(V^*, E^*)$ is a 3-connected graph obtained after a set of contractions on G such that for every $e^*{\in}E^*$, $\mathrm{w}(e^*){\le}\lceil|V|/3\rceil{-}2$, then there exists a set $\{u, v, w\}{\in}V^*$ and a partition of V^* into (V_1^*, V_2^*) such that:*

1. *$V_1^*{\cap}V_2^*{=}\emptyset$ and $V_1^*{\cup}V_2^*{=}V^*$,*
2. *$G^*[V_1^*]$ and $G^*[V_2^*]$ are connected,*
3. *$\{u, w\}, \{v, w\}{\in}E^*$,*
4. *$w{\in}V_1^*$, $u, v{\in}V_2^*$, and u, v are not cutpoints of $G^*[V_2^*]$,*
5. *$\mathrm{w}(V_2^*){\le}\lfloor|V|/3\rfloor{+}1$.*

Moreover, such a partition and $\{u, v, w\}$ can be found in $O(|E|)$ time.

Proof. If G^* has a triangle $\{u, v, w\}$ then since G^* is 3-connected, $V_2^*{=}\{u, v\}$ and $V_1^*{=}V^*\setminus V_2^*$ is a good partition. So assume G^* is a triangle-free graph and therefore $|V^*|{>}3$.

Assume that there exists a (triangle-free) counterexample graph G^* that does not have a partition with the desired properties. We will derive a contradiction. Following Theorem 2.2, we can find a non-separating ear decomposition $G^*{=}P_0^*{\cup}P_1^* \cdots {\cup}P_k^*$. A property of their construction that we will use is that if $|P_i^*|{>}3$, then each internal node of P_i^* has degree two in G_i^* (as defined in Section 2). We will show a series of properties for the graph G^* and its no-separating ear decomposition, leading eventually to a contradiction.

Claim 1. For any $0{\le}i{<}k$, if G_i^* has two adjacent degree-2 nodes, then $\mathrm{w}(G_i^*){\le}\lfloor 2|V|/3\rfloor$. In particular, $\mathrm{w}(P_0^*){\le}\lfloor 2|V|/3\rfloor$, and if $|P_i^*|{>}3$ then $\mathrm{w}(G_i^*){\le}\lfloor 2|V|/3\rfloor$.

Proof. Suppose that $\mathrm{w}(G_i^*){\ge}\lfloor 2|V|/3\rfloor{+}1$ and that G_i^* has two adjacent degree-2 nodes w, v. Since G^* is 3-connected, w and v are adjacent respectively to some nodes u, z of $\overline{G_i^*}$. Note that $u{\ne}z$ because G^* does not have a triangle. If u is not a cutpoint of $\overline{G_i^*}$, set $V_1^*{=}G_i^*\setminus\{v\}$ and $V_2^*{=}V^*\setminus V_1^*$. From the properties of the non-separating ear decomposition, it is easy to see that (V_1^*, V_2^*) is a good partition: $G[V_1^*]$ is connected, since G_i^* is biconnected, $G[V_2^*]$ is connected since $\overline{G_i^*}$ is connected, u, v are not cutpoints of $G[V_2^*]$, and $|V_2^*|{\le}|V|{-}\lfloor 2|V|/3\rfloor{=}\lceil|V|/3\rceil$. Otherwise, if u is a cutpoint of $\overline{G_i^*}$, let H^* be the connected component of $\overline{G_i^*}\setminus\{u\}$ that contains node z. Set $V_2^*{=}H^*{\cup}\{u, v\}$ and $V_1^*{=}V^*\setminus V_2^*$. It is easy to see again that this is a good partition. Notice that since $G^*\setminus\{u, v\}$ is connected, therefore $G^*[V_1^*]$ is also connected.

The nodes of P_0^* have degree 2 in G_0^*, hence the claim holds for P_0^*, and more generally for any i such that $|P_i^*|{>}3$, since the internal nodes of P_i^* have degree 2 in G_i^*. □

We consider now the first ear P_1^* and the corresponding graph G_1^*.

Claim 2. G_1^* consists of three parallel disjoint paths of length at least 3 between two nodes, and $\mathrm{w}(G_1^*){\le}\lfloor 2|V|/3\rfloor$.

Proof. Suppose first that $|P_1^*|{=}3$, and let $P_1^*{=}u, w, v$. At least one of the two $u{-}v$ paths of the cycle P_0^* has weight at most $\mathrm{w}(P_0^*)/2{+}1$. Let V_2^* be this path and $V_1^*{=}V^*\setminus V_2^*$. Since $\mathrm{w}(P_0^*){\le}\lfloor 2|V|/3\rfloor$, it follows that $\mathrm{w}(V_2^*){\le}\lfloor|V|/3\rfloor{+}1$. Clearly $G[V_1^*]$ and $G[V_2^*]$ are connected, and all the desired properties are satisfied, so this is a good partition.

Therefore, $|P_1^*|{>}3$. All the internal nodes of P_1^* have degree 2. By Claim 1, $\mathrm{w}(G_1^*){\le}\lfloor 2|V|/3\rfloor$, and G_1^* consists of three disjoint paths between two nodes x, y, the endpoints of the ear P_1^*. If one of the paths is an edge (x, y), then let Q be the lighter of the other two paths. Set $w{=}x$, $u{=}y$, let v be the node of Q adjacent to x, $V_2^*{=}Q\setminus\{x\}$, and $V_1^*{=}V^*\setminus V_2^*$; it is easy to see that this is a good partition. If one of the paths has length 2, then let Q again be the lighter of the other two paths.

Setting u=x, v=y, w the internal node of Q, V_2^*=Q, V_1^*=$V^*\setminus V_2^*$ gives a good partition. Note that w(Q)≤(w(G_1^*)+1)/2≤(⌊2|V|/3⌋+1)/2, hence w(Q)≤⌈|V|/3⌉.

We conclude that G_1^* satisfies the claim. □

Thus G_1^* consists of three paths of length at least 3 between two nodes x, y, and w(G_1^*)≤⌊2|V|/3⌋. We consider now the next ear P_2^*.

Claim 3. $|P_2^*|>3$ and w(G_2^*)≤⌊2|V|/3⌋.

Proof. Suppose that $|P_2^*|=3$ and let $P_2^*=u, w, v$. Let R_1, R_2 be two disjoint paths in G_1^* connecting u and v, and assume without loss of generality that w(R_2)≤w(R_1). Set V_2^*=R_2 and V_1^*=$V^*\setminus V_2^*$. Note that every degree-2 node of $G_1^*-R_2$ has an edge to $\overline{G_1^*}$, and if x or y is not in R_2 then it is adjacent to at least one degree-2 node of $G_1^*-R_2$. Hence $G[V_2^*]$ is connected. Also, w(R_2)≤w(G_1^*)/2≤|V|/3.

We conclude that $|P_2^*|>3$, and hence by Claim 1, w(G_2^*)≤⌊2|V|/3⌋. □

By Claim 2, G_1^* consists of three paths of length at least 3 between two nodes x, y. Since $|P_2^*|>3$ (by Claim 3), all internal nodes of P_2^* have degree 2 in G_2^*. The endpoints of the path P_2^* are either internal nodes of different paths of G_1^* (in which case G_2^* is homeomorphic to K_4, the complete graph on 4 nodes), or they both lie on one of the three paths of G_1^* (either or both endpoints may coincide with the degree-3 nodes x, y of G_1^*). The graph G_2^* is a planar graph in either case. Consider a planar embedding of G_2^*. It has four faces. The sum of the weights of the four faces is 2w(G_2^*)+4 (every edge is counted twice and every node is counted as many times as its degree). Therefore at least one of the faces has weight at most w(G_2^*)/2+1≤(|V|/3)+1. Let C be the bounding cycle of such a face. If the cycle C has a chord (the chord of course must be embedded outside the face), then let (w, u) be a chord such that one of the paths of C connecting w to u is chordless, let R be this path, and let v be its node adjacent to w. Set V_2^*=$R\setminus\{w\}$ and V_1^*=$V^*\setminus V_2^*$. Then w(V_2^*)≤⌊|V|/3⌋, and it is easy to check also that $G[V_1^*]$ is connected. If C is chordless, then let u, w, v be three consecutive nodes of C. Set V_2^*=$C\setminus\{w\}$ and V_1^*=$V^*\setminus V_2^*$. Again, it is easy to check that the partition satisfies the required properties of the lemma.

This concludes the proof of the lemma. ■

Using Lemma 4.15, then Lemma 4.11, and the idea of the proof for Theorem 4.8, we can prove that when G is 2-connected and all $p(i)$=±1, the DBCP problem has a solution for c_p=1 and c_s=2. We find a suitable convex embedding of the 3-connected graph G^* using Lemma 4.17 and Lemma 4.7, and then embed the nodes of the contracted pseudo-paths appropriately along the segments corresponding to the weighted edges. Some care is needed to carry out the argument as in the proof for Theorem 4.8, since as the line tangent to the circle rotates, the order of the projections of many nodes may change at once, namely the nodes on an edge perpendicular to the rotating line.

Theorem 4.18. *If G is 2-connected, $\forall i, p(i)$=±1, then the DBCP problem has a solution for c_p=1 and c_s=2. Moreover, this solution can be found in randomized polynomial time.*

Proof. Using Lemma 4.15, we consider two cases:

(i) There is a separation pair $\{u, v\}\in V$ such that if $H_1, \ldots, H_k$ are the connected components of $G\setminus\{u, v\}$, for all i, $|V_{H_i}|$<⌊2|V|/3⌋. In this case Lemma 4.11 proves the theorem.
(ii) After a set of contractions, G can be transformed into a 3-connected graph G^*=(V^*, E^*) with weighted edges such that for any edge $e^*\in E^*$, w(e^*)≤⌈|V|/3⌉−2. In this case the proof is similar to the proof of Theorem 4.8. Notice that if G^* contains a triangle then the proof is much simpler as in the proof of Lemma 4.5 but here to avoid repetition, we use the approach in the proof of Theorem 4.8 and prove the theorem once for all cases of G^*.

Using Lemma 4.17, we can find $\{u, v, w\} \in V^*$ and a partition (V_1^*, V_2^*) of V^* with properties described in the lemma. Set $X=\{u, v, w\}$. Using Lemma 4.7, G^* has a convex X-embedding in general position, $f^*:V^* \to \mathbb{R}^2$, as described in the lemma and depicted in Fig. 3. Now, from this embedding, we get a convex X-embedding $f:V \to \mathbb{R}^2$ for G as follows. For any $i \in V \cap V^*$, $f(i)=f^*(i)$. For any edge $\{i, j\} \in E^*$ such that $\{i, j\}$ represents an induced subgraph of G, we represent it by a pseudo-path P of G between i and j and place the nodes of P in order on random places on the line segment that connects $f(i)$ to $f(j)$. If the edge $\{i, j\} \in E^*$ is between a node in V_1^* and a node in V_2^* and represents a pseudo-path P in G , we place the nodes in P in order on random places on the segment that connects $f(i)$ to $f(j)$ but above the line $\mathcal{L}_1$. Hence, by this process, we get a convex X-embedding for G which is in general position (almost surely) except for the nodes that are part of a pseudo-path. From Lemma 4.7, the embedding has the following property. Consider any line on the plane, and the subset of nodes whose points lie on the same side of the line. If the subset has size at least $\lfloor |V|/3 \rfloor + 1$ then it induces a connected subgraph of G.

The rest of the proof is similar to the proof of Theorem 4.8. We consider again a circle C around $f(u), f(v), f(w)$ in $\mathbb{R}^2$ as shown in Fig. 3. Also consider a directed line $\mathcal{L}$ tangent to the circle C at point A and project the nodes of G onto the line $\mathcal{L}$ (we consider a line such that the projections are distinct). We label nodes based on their projection order on the line $\mathcal{L}$ from left to right from $1^{(\mathcal{L})}$ to $|V|^{(\mathcal{L})}$. For each $t=1, \ldots, |V|$, let $V^{(\mathcal{L})}(t)=\{1^{(\mathcal{L})}, \ldots, t^{(\mathcal{L})}\}$ denote the set of the first t nodes in this ordering. Since the embedding f has the properties described in Lemma 4.17 and 4.7, for all $\lfloor |V|/3 \rfloor + 1 \le t \le \lceil 2|V|/3 \rceil - 1$, if we set $V_1=V^{(\mathcal{L})}(t)$ and $V_2=V \setminus V_1$, then $G[V_1]$ and $G[V_2]$ are both connected. Define $V_1'=V^{(\mathcal{L})}(\lfloor |V|/3 \rfloor)$ and $V_2'=V \setminus V^{(\mathcal{L})}(\lceil 2|V|/3 \rceil)$, i.e., V_1' contains the first $\lfloor |V|/3 \rfloor$ nodes and V_2' the last $\lfloor |V|/3 \rfloor$ nodes in the ordering. If $p(V_1')p(V_2') \ge 0$, then there must exist an index j with $\lfloor |V|/3 \rfloor \le j \le \lceil 2|V|/3 \rceil$ such that $p(V^{(\mathcal{L})}(j))=\sum_{i=1}^{j} p(i^{(\mathcal{L})})=0$. Consequently, there is an index t such that $\lfloor |V|/3 \rfloor + 1 \le t \le \lceil 2|V|/3 \rceil - 1$ and $|p(V^{(\mathcal{L})}(t))| \le 1$: if $j=\lfloor |V|/3 \rfloor$ then let $t=j+1$, if $j=\lceil 2|V|/3 \rceil$ then $t=j-1$, and otherwise let $t=j$. Hence, $V_1=V^{(\mathcal{L})}(t)=\{1^{(\mathcal{L})}, \ldots, t^{(\mathcal{L})}\}$ and $V_2=V \setminus V_1$ is a good partition. Therefore, if $p(V_1')p(V_2') \ge 0$ then we can obtain a good partition. We will show that there is a line such that $p(V_1')p(V_2') \ge 0$.

Assume without loss of generality that in the initial position of the line, $p(V_1')>0$ and $p(V_2')<0$. As we move $\mathcal{L}$ from being tangent at point A to point B where AB is a diameter of the circle C, and consider the resulting partition at point B, since V_1' and V_2' completely switch places compared to the partition at point A, at point B we have $p(V_1')<0$ and $p(V_2')>0$. Hence, as we move $\mathcal{L}$ from being tangent at point A to point B and keep it tangent to the circle, in the resulting partitions, $p(V_1')$ goes at some point from some positive value to a nonpositive value. Notice that the ordering of the projections on the line changes only if $\mathcal{L}$ passes a point D on the circle such that at D, $\mathcal{L}$ is perpendicular to a line that connects $f(i)$ to $f(j)$ for $i, j \in V^*$; then the order of i and j is switched, and if (i, j) is an edge of G^* then also the ordering of the nodes in the corresponding pseudopath is reversed. Note that all of these affected nodes are consecutive in the ordering. If $p(V_1')$ changes then V_1' must contain at least one of the affected nodes, and similarly for $p(V_2')$.

So assume $p(V_1')>0$ in a projection on a line $\mathcal{L}_b$ slightly before a perpendicular point and $p(V_1') \le 0$ in a resulting projection on a line $\mathcal{L}_a$ slightly after a perpendicular point. Define $V_{1b}':=\{1^{(\mathcal{L}_b)}, \ldots, \lfloor |V|/3 \rfloor^{(\mathcal{L}_b)}\}$, $V_{2b}':=\{(|V|-\lfloor |V|/3 \rfloor+1)^{(\mathcal{L}_b)}, \ldots, |V|^{(\mathcal{L}_b)}\}$, $V_{1a}':=\{1^{(\mathcal{L}_a)}, \ldots, \lfloor |V|/3 \rfloor^{(\mathcal{L}_a)}\}$, and $V_{2a}':=\{(|V|-\lfloor |V|/3 \rfloor+1)^{(\mathcal{L}_a)}, \ldots, |V|^{(\mathcal{L}_a)}\}$. So $p(V_{1b}')>0$ and $p(V_{1a}') \le 0$.

If $p(V'_{1b})p(V'_{2b}){\geq}0$, as we argued previously, we can find a good partition (V_1, V_2) such that $V'_{1b}{\subseteq}V_1$ and $V'_{2b}{\subseteq}V_2$, and we are done. So assume $p(V'_{1b}){>}0$ and $p(V'_{2b}){<}0$. Since for any $e^*{\in}E^*$, $\mathrm{w}(e^*){\leq}\lceil|V|/3\rceil{-}2$, the ordering of the nodes based on the projections on lines $\mathcal{L}_b$ and $\mathcal{L}_a$ is different for at most $\lceil|V|/3\rceil$ consecutive nodes (the ordering of at most $\lceil|V|/3\rceil$ consecutive nodes reverses as we move from $\mathcal{L}_b$ to $\mathcal{L}_a$). Since $V'_{1b}{\neq}V'_{1a}$ (recall $p(V'_{1b}){>}0$ and $p(V'_{1a}){\leq}0$), the set of reversed nodes includes one of the first $\lfloor|V|/3\rfloor$ nodes in the ordering, hence it cannot extend to include also one of the last $\lfloor|V|/3\rfloor$ nodes; therefore $p(V'_{2a}){=}p(V'_{2b}){<}0$. Thus, $p(V'_{1a})p(V'_{2a}){\geq}0$ and hence, as we argued before, there is a good partition (V_1, V_2) such that $V'_{1a}{\subseteq}V_1$ and $V'_{2a}{\subseteq}V_2$.

Regarding the computation of a good partition, after we compute the contracted graph G^* and its convex embedding, the rest of the computation can be easily carried out in $O(|V|^2 \log |V|)$ time.

■

Similar to Corollary 4.9, the approach used in the proof of Theorem 4.18, can also be used for the case when the weights are arbitrary (not necessarily ±1) and $p(V){\neq}0$. It is easy to verify using similar arguments that in this case, if G is 2-connected, the DBCP problem has a connected partition (V_1, V_2) such that $|p(V_1){-}p(V)/2|, |p(V_2){-}p(V)/2|{\leq}\max_{j\in V} |p(j)|$ and $|V_1|, |V_2|{\geq}\lceil|V|/3\rceil$.

Corollary 4.19. *If G is 2-connected, then the DBCP problem (with general p and not necessarily satisfying $p(V){=}0$) has a solution for $c_p{=}\max_{j\in V} |p(j)|$ and $c_s{=}2$. Moreover, this solution can be found in randomized polynomial time.*

5 GRAPHS WITH TWO TYPES OF NODES

Assume G is a connected graph with nodes colored either red ($R{\subseteq}V$) or blue ($B{\subseteq}V$). Let $|V|{=}n$, $|R|{=}n_r$, and $|B|{=}n_b$. If G is 3-connected, set $p(i){=}1$ if $i{\in}R$ and $p(i){=}{-}1$ if $i{\in}B$. Corollary 4.9 implies then that there is always a connected partition (V_1, V_2) of V that splits both the blue and the red nodes evenly (assuming n_r and n_b are both even), i.e., such that $|V_1|{=}|V_2|$, $|R{\cap}V_1|{=}|R{\cap}V_2|$, and $|B{\cap}V_1|{=}|B{\cap}V_2|$. (If n_r and/or n_b are not even, then one side will contain one more red or blue node.)

Corollary 5.1. *Given a 3-connected graph G with nodes colored either red ($R{\subseteq}V$) or blue ($B{\subseteq}V$). There is always a partition (V_1, V_2) of V such that $G[V_1]$ and $G[V_2]$ are connected, $|V_1|{=}|V_2|$, $|R{\cap}V_1|{=}|R{\cap}V_2|$, and $|B{\cap}V_1|{=}|B{\cap}V_2|$ (assuming $|R|$ and $|B|$ are both even). Such a partition can be computed in randomized polynomial time.*

Proof. Suppose without loss of generality that $n_r{\geq}n_b$ and let $n_r{-}n_b{=}2t$ and $n_r{+}n_b{=}n{=}2m$. Set $p(i){=}1$ for $i{\in}R$ and $p(i){=}{-}1$ for $i{\in}B$. Then $p(V){=}2t$. From the equations, we have $n_r{=}m{+}t$ and $n_b{=}m{-}t$.

From Corollary 4.9 we can find a partition (V_1, V_2) such that $|V_1|{=}|V_2|$ and $|p(V_1){-}p(V)/2|, |p(V_1){-}p(V)/2|{\leq}1$. Let $r_1{=}|R{\cap}V_1|$ and $b_1{=}|B{\cap}V_1|$. We have $r_1{+}b_1{=}n/2{=}m$ and $t{-}1{\leq}r_1{-}b_1{\leq}t{+}1$. Therefore, $(m{+}t)/2{-}(1/2){\leq}r_1{\leq}(m{+}t)/2{+}(1/2)$. Since r_1 is an integer and $n_r{=}m{+}t$ is even, it follows that $r_1{=}(m{+}t)/2{=}n_r/2$. Hence, $b_1{=}(m{-}t)/2{=}n_b/2$. Therefore, V_2 also contains $n_r/2$ red nodes and $n_b/2$ blue nodes. ■

If G is only 2-connected, we may not always get a perfect partition. Assume wlog that $n_r{\leq}n_b$. If for every $v{\in}R$ and $u{\in}B$, we set $p(v){=}1$ and $p(u){=}{-}n_r/n_b$, Corollary 4.19 implies that there is always a connected partition (V_1, V_2) of V such that both $|(|R{\cap}V_1|{-}n_r/n_b|B{\cap}V_1|)|{\leq}1$ and $|(|R{\cap}V_2|{-}n_r/n_b|B{\cap}V_2|)|{\leq}1$, and also $\max\{\frac{|V_1|}{|V_2|}, \frac{|V_2|}{|V_1|}\}{\leq}2$. Thus, the ratio of red to blue nodes in each side V_i differs from the ratio n_r/n_b in the whole graph by $O(1/n)$. Hence if the numbers of red and blue nodes are $\omega(1)$, then the two types are presented in both sides of the partition in approximately the same proportion as in the whole graph.

Corollary 5.2. *Given a 2-connected graph G with nodes colored either red ($R \subseteq V$) or blue ($B \subseteq V$), and assume wlog $|R| \le |B|$. We can always find in randomized polynomial time a partition (V_1, V_2) of V such that $G[V_1]$ and $G[V_2]$ are connected, $|V_1|, |V_2| \ge \lceil |V|/3 \rceil$, and the ratio of red to blue nodes in each side V_i differs from the ratio $|R|/|B|$ in the whole graph by $O(1/n)$.*

A similar result as Corollary 5.1 has been previously provided by Nagamochi et al. [15]. However, Corollary 5.2 extends that result to 2-connected graphs.

6 CONCLUSION

In this paper, we introduced and studied the problem of partitioning a graph into two connected subgraphs that satisfy simultaneouly two objectives: (1) they balance the supply and demand within each side of the partition (or more generally, for the case of $p(V) \neq 0$, they split approximately equally the excess supply/demand between the two sides), and (2) the two sides are large and have roughly comparable size (they are both $\Omega(|V|)$). We showed that for 2-connected graphs it is always possible to achieve both objectives at the same time, and for 3-connected graphs there is a partition that is essentially perfectly balanced in both objectives. Furthermore, these partitions can be computed in polynomial time. This is a paradigmatic bi-objective balancing problem. We observed how it can be easily used to find a connected partition of a graph with two types of nodes that is balanced with respect to the sizes of both types. Overall, we believe that the novel techniques used in this paper can be applied to partitioning heterogeneous networks in various contexts.

There are several interesting further directions that suggest themselves. First, extend the theory and algorithms to find doubly balanced connected partitions to more than two parts. Second, even considering only the supply/demand objective, does the analogue of the results of Lováz and Gyori [9, 14] for the connected k-way partitioning of k-connected graphs with respect to size (which corresponds to $p(i)=1$) extend to the supply/demand case ($p(i)=\pm 1$) for $k>3$? And is there a polynomial algorithm that constructs such a partition? Finally, extend the results of Section 5 to graphs with more than two types of nodes, that is, can we partition (under suitable conditions) a graph with several types of nodes to two (or more) large connected subgraphs that preserve approximately the diversity (the proportions of the types) of the whole population? (Bisecting 4-connected graphs with three types of nodes has already been studied by Ishii et al. [11].)

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