EVALUATING THE TOPOLOGICAL ROBUSTNESS OF POWER GRIDS TO LINE FAILURES

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Summary

We use the *mutual edge flow change ratios* (the ratio between the change of flow on an edge, and the initial flow on the failed edge) to evaluate the topological robustness of power grids to line failures. In particular, we show that mutual edge flow change ratios are independent of the power supply/demand distribution and solely depend on the grid structure. Then, we define and analytically compute the *failure cost of an edge* and the *average edge failure cost in a graph*, and demonstrate that the results can be used to study the robustness of power grids to a single line failure.

Model

We adopt the linearized (or DC) power flow model, which is widely used as an approximation for the AC power flow model [1,4]. We represent the power grid by an undirected graph $G = (V, E)$ where V and E correspond to the buses and transmission lines, respectively. p_v is the active power *supply* ($p_v > 0$) or *demand* ($p_v < 0$) at node $v \in V$ (for a *neutral node* $p_v = 0$). We assume *pure reactive* lines, where each edge $\{u, v\}$ is characterized by its *reactance* $x_{uv} = x_{vu}$. A *power flow* is a solution (f, θ) of:

$$\sum_{v \in N(u)} f_{uv} = p_u, \ \forall\ u \in V \qquad (1)$$

$$\theta_u - \theta_v - x_{uv} f_{uv} = 0, \ \forall\ \{u, v\} \in E \qquad (2)$$

where $N(u)$ is the set of neighbors of node u, f_{uv} is the power flow from node u to node v, and θ_u is the phase angle of node u. Eq.(1)-(2) are equivalent to the matrix equation: $A\Theta = P$, where $\Theta \in \mathbb{R}^{|V|\times 1}$ is the vector of phase angles, $P \in \mathbb{R}^{|V|\times 1}$ is the power supply/demand vector, and $A = [a_{ij}] \in \mathbb{R}^{|V|\times|V|}$ is the *admittance matrix* of the graph G. The power flow equations can be solved by using the *Moore-Penrose Pseudo-inverse* of the admittance matrix, $A^+ = [a^+_{ij}]$ [2].

To study the effects of a *single edge (e') failure*, we define the ratio between the change of flow on an edge, e, and the initial flow on the failed edge, e', as *mutual edge flow change ratio*: $M_{e,e'} = |\Delta f_e / f_{e'}|$. The mutual edge flow change ratio corresponds to the Line Outage Distribution Factor (LODF) defined in [4, P. 307].

Failure Impact

The following theorem provides an analytical rank-1 update of the pseudo-inverse of the admittance matrix.

Theorem 1. *If $\{i, j\}$ is not a cut-edge, then,*

$$A'^+ = (A + a_{ij} X X^t)^+ = A^+ - \frac{1}{a_{ij}^{-1} + X^t A^+ X} A^+ X X^t A^+$$

in which X is an $n \times 1$ vector with 1 in i^{th} entry, -1 in j^{th} entry, and 0 elsewhere.

Corollary 1. *The flow on an edge $\{r, s\}$ after a failure in the non-cut-edge $\{i, j\}$ is,*

$$f'_{rs} = f_{rs} - \frac{a_{rs}}{a_{ij}} \frac{(a^+_{ri} - a^+_{rj}) - (a^+_{si} - a^+_{sj})}{a_{ij}^{-1} - 2(a^+)_{ij} + (a^+)_{ii} + (a^+)_{jj}} f_{ij}.$$

To focus solely on topological robustness, in this abstract we assume that $x_{uv} = 1\ \forall \{u, v\} \in E$. In this case, the admittance matrix A is the *Laplacian matrix* of the graph and using Corollary 1 the mutual edge flow change ratios can computed as follows.

Lemma 1. *The mutual edge flow change ratio for an edge $e = \{r, s\} \in E$ after a failure in a non-cut-edge $e' = \{i, j\} \in E$ is,*

$$M_{e,e'} = \left| \frac{(a^+_{ri} - a^+_{rj}) - (a^+_{si} - a^+_{sj})}{-1 - 2(a^+)_{ij} + (a^+)_{ii} + (a^+)_{jj}} \right|.$$

The Lemma implies that the mutual edge flow change ratios are independent of the power supply/demand distribution and solely depend on the grid structure.

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Network Robustness

Definition. *The* failure cost *of an edge e in G is denoted by FC_e and defined as follows:* $FC_e := \frac{1}{m-1}\sum_{\substack{e'\in E \\ e'\neq e}}(M_{e',e})^2$.

The failure cost of an edge e is a good measure of the average changes that occur in the flows of the other edges as a result of the failure in an edge e. Determining the costs can help constructing a reliable power grid in two ways: (i) by designing networks with a minimum maximum failure cost, and (ii) by setting the power supply and demand values such that edges with high failure costs carry small flows. The following Lemma analytically shows the relation between the failure cost of a non-cut-edge and the *resistance distance* between its end nodes. The resistance distance between two nodes $i, j \in V$ is $r(i,j) := a^+_{ii} + a^+_{jj} - 2a^+_{ij}$.

Lemma 2. *In a connected graph G, for any non-cut-edge $e = \{i, j\}$,*

$$FC_e = \frac{1}{m-1}\frac{r(i,j)}{1-r(i,j)}. \tag{3}$$

Eq. (3) is very insightful. Intuitively, it demonstrates that failures in edges with high resistance distance values have a strong effect on the other edges. Moreover, (3) allows to obtain a bound on the average edge failure cost, which is defined below as a metric for the robustness of a graph to a single edge failure.

Definition. *In a graph G with n nodes and m edges, the* average edge failure cost *is defined as,* $\overline{FC}_G := \frac{1}{m}\sum_{e\in E} FC_e$.

Using (3), the following Lemma provides a lower bound on the average edge failure cost in a graph.

Lemma 3. *In a 2-edge-connected graph G,*

$$\frac{1}{m}\Big(\frac{m-1}{n-1} - \frac{m-1}{m}\Big)^{-1} \leq \overline{FC}_G, \tag{4}$$

and equality holds, if for any two edges $e = \{i, j\}$ and $e' = \{p, q\}$, $r(i, j) = r(p, q)$.

Corollary 2. *In a symmetric graph G, $\overline{FC}_G = (\frac{m^2-m}{n-1} - (m-1))^{-1}$. Moreover, for any graph H with the same number of nodes and edges as G, $\overline{FC}_H \geq \overline{FC}_G$.*

Corollary 2 demonstrates that symmetric graphs have the lowest average edge failure cost among all the graphs with the same number of nodes and edges. Moreover, from Lemma 3 and Corollary 2 it can be concluded that as graphs become more symmetrical, their average edge failure cost ($\overline{FC}_G$) decreases. To demonstrate this numerically, Fig. 1 shows the average edge failure cost of the graph ($\overline{FC}_G$) and the maximum mutual edge flow change ratio ($\max_{e,e'\in E} M_{e,e'}$) versus the probability of rewiring (p) in Watts and Strogatz graphs [3] with 30 nodes and 60 edges. Initially ($p = 0$), G is a 4-regular graph (namely, every node is connected to exactly 4 other nodes). However, as p increases, G tends toward a random graph with no symmetry. Thus, an increase in p in the Watts and Strogatz graph can be considered as decrease in the symmetry of the graph. As expected, the figure shows that as p increases, both the average edge failure cost of the graph ($\overline{FC}_G$) and the maximum mutual edge flow change ratio ($\max_{e,e'\in E} M_{e,e'}$) increase.

Overall, the results suggest that as graphs become more symmetrical, they become more robust against single edge failures.

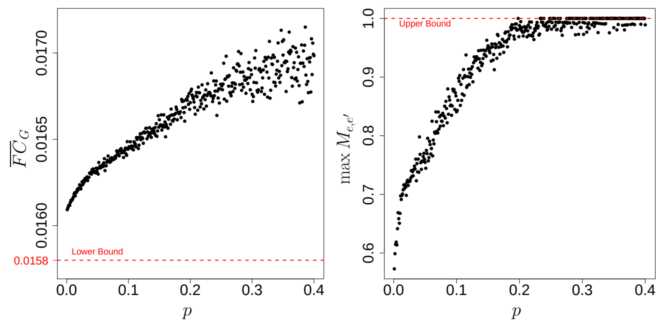

Figure 1: The average edge failure cost of the graph ($\overline{FC}_G$) and the maximum mutual edge flow change ratio ($\max_{e,e'\in E} M_{e,e'}$) versus the probability of rewiring (p) in a Watts and Strogatz graph with 30 nodes and 60 edges. Each point is the average over 100 generated graphs with the same parameters.

References

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