Doubly Balanced Connected Graph Partitioning

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Motivation

- *Power Grid Islanding* → mitigate cascading failures
- Partition the network into smaller operable *islands*
  - *Supply = demand* in each island
  - Island is *large enough* to have the capacity to deliver power
- *Doubly Balanced Connected graph Partitioning (DBCP):*

Given: Connected graph $G = (V, E)$ with a weight (supply/demand) function $p: V \to \mathbb{Z}$ satisfying $p(V) = \sum_{j \in V} p(j) = 0$

Objective: Partition V into (V_1, V_2) such that:

1. $G[V_1]$ and $G[V_2]$ are connected
2. $|p(V_1)|, |p(V_2)| \leq c_p$ for some constant c_p
3. $\max\left\{\frac{|V_1|}{|V_2|}, \frac{|V_2|}{|V_1|}\right\} \leq c_s$ for some constant c_s

Presentation Outline

- Related Work
- Balancing a single objective ($|p(V_1)|, |p(V_2)| \leq c_p$)
- Balancing both objectives
  - 3-connected graphs
  - 2-connected graphs
- Graphs with two types of Nodes

Related Work

Theorem (Lovász and Gyori 1977). Let $G = (V, E)$ be a k-connected graph. Let $n = |V|$, $v_1, v_2, \dots, v_k \in V$, and $n_1, n_2, \dots, n_k$ be positive integers satisfying $n_1 + n_2 + \cdots + n_k = n$. Then, there exists a partition of V into $(V_1, V_2, \dots, V_k)$ satisfying $v_i \in V_i$, $|V_i| = n_i$, and $G[V_i]$ is connected for $i = 1,2, \dots, k$.

- For $k > 3$ no polynomial time algorithm is known to find such partition

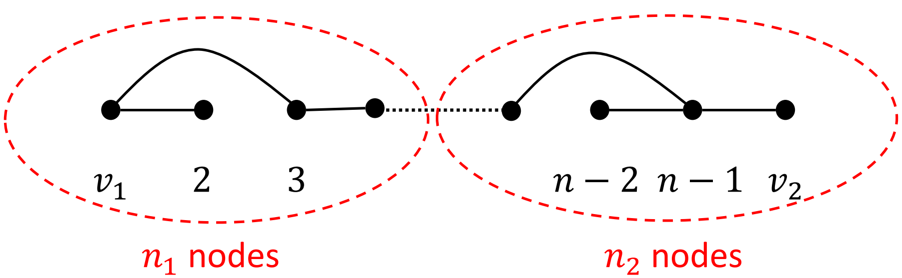
st-numbering

- For $k = 2$, use *st-numbering:*

Given nodes $\{s, t\}$ in a graph G
An *st-numbering* is numbering for nodes such that:

1. Nodes of G are numbered from 1 to n
2. Node s receives number 1 and node t receives number n
3. Every node except s and t is adjacent both to a lower-numbered and to a higher-numbered node

(Evans and Tarjan 1976). An st-numbering for a 2-connected graph G can be found in $O(|V| + |E|)$ for any pair of node.

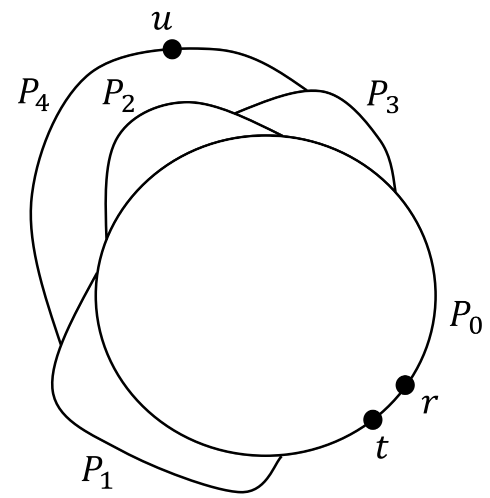


Nonseparating Ear Decomposition

- For $k = 3$, use *nonseparating ear decomposition*

Let H be a subgraph of a graph G
An *ear* of H in G is a nontrivial path in G whose ends lie in H but whose internal nodes do not
An ear decomposition of G is a decomposition $G = P_0 \cup \cdots \cup P_k$ such that:

1. P_0 is a cycle
2. P_i for $i \geq 1$ is an ear of $P_0 \cup P_1 \cup \cdots \cup P_{i-1}$



Every 2-connected graph has an ear decomposition (and vice-versa), and such a decomposition can be found in linear time.

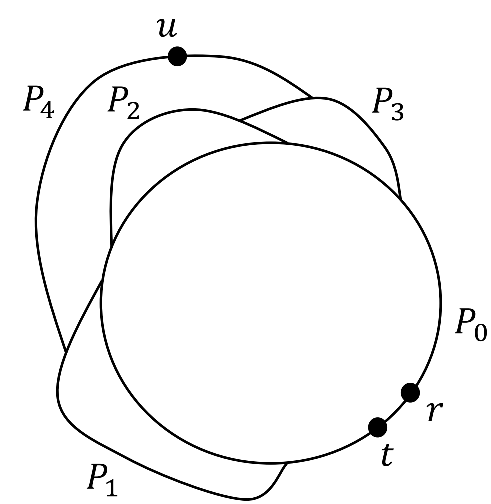
An ear decomposition is through edge $\{t, r\}$ and avoiding vertex u:

1. Cycle P_0 contains edge $\{t, r\}$
2. The last nontrivial ear, has u as its only internal vertex

Nonseparating Ear Decomposition

Let $V_i = V(P_0) \cup V(P_1) \cup \cdots \cup V(P_i)$
Let $G_i = G[V_i]$ and $\bar{G}_i = G[V \backslash V_i]$
A nonseparating ear decomposition is an ear decomposition such that for all $0 \leq i < k$:

1. Graph $\bar{G}_i$ is connected
2. Each internal vertex of P_i has a neighbor in $\bar{G}_i$



(Cheriyan and Maheshwari 1988). Given an edge $\{t, r\}$ and a vertex u of a 3-connected graph G, a nonseparating ear decomposition of G through $\{t, r\}$ and avoiding u can be found in $O(|V| + |E|)$ time.

- Using *nonseparating ear decomposition for* $k = 3$, a solution can be found for the Lováz/Gyori theorem

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- Balancing both objectives
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  - ✓ 2-connected graphs
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Balancing Supply and Demand Only

Given: Connected graph $G = (V, E)$ with a weight (supply/demand) function $p: V \rightarrow \mathbb{Z}$ satisfying $p(V) = \sum_{j \in V} p(j) = 0$

Objective: Partition V into (V_1, V_2) such that:

1. $G[V_1]$ and $G[V_2]$ are connected
2. $|p(V_1)|, |p(V_2)| \leq c_p$ for some constant c_p

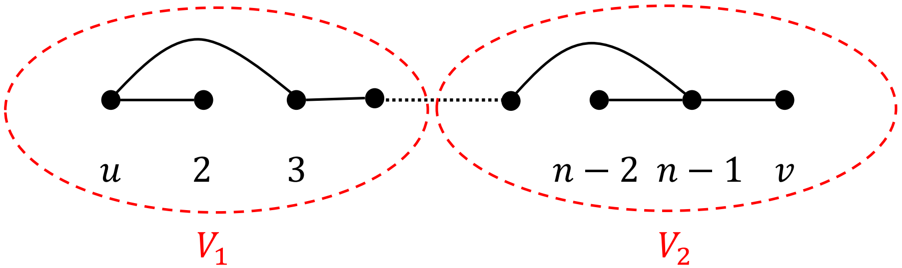
Proposition. It is strongly NP-hard to determine whether there is a solution to problem above for $c_p = 0$, even when G in 2-connected.

Proposition. If G is not 2-connected, then this problem is NP-hard even when $\forall\, i, p(i) = \pm 1$.

Balancing Supply and Demand Only

Proposition. If G is 2-connected, there is always a solution such that $|p(V_1)|, |p(V_2)| \leq \max_{j \in V} \frac{|p(j)|}{2}$, and can be found in polynomial time.

Proof. Use st-numbering between nodes u, v with $p(u)p(v) > 0$.



There is an i such that $\sum_{j=1}^{i} p(j) > 0$ and $\sum_{j=1}^{i+1} p(j) \leq 0$.

- If $\forall\, i, p(i) = \pm 1$ and G is 2-connected, there is always a solution with $p(V_1) = p(V_2) = 0$.

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Doubly Balanced Connected Partitioning

Given: Connected graph $G = (V, E)$ with a weight (supply/demand) function $p: V \to \mathbb{Z}$ satisfying $p(V) = \sum_{j \in V} p(j) = 0$

Objective: Partition V into (V_1, V_2) such that:

1. $G[V_1]$ and $G[V_2]$ are connected
2. $|p(V_1)|, |p(V_2)| \leq c_p$ for some constant c_p
3. $\max\left\{\frac{|V_1|}{|V_2|}, \frac{|V_2|}{|V_1|}\right\} \leq c_s$ for some constant c_s

- We assume that $\forall\, i, p(i) = \pm 1$
- Techniques can be used in general case as well

3-Connected with a triangle

Lemma. If G is a 3-connected graph *with a triangle*, then

1. If $|V| \equiv 0 \pmod 4$, then there exists a solution to the DBCP problem with $p(V_1) = p(V_2) = 0$ and $|V_1| = |V_2|$.
2. If $|V| \equiv 2 \pmod 4$, then there exists a solution to the DBCP problem with $p(V_1) = p(V_2) = 0$ and $|V_1| = |V_2| + 2$.

Proof. Use convex embedding of graphs

Let $X \subset V$. A convex X-embedding of G is any mapping $f: V \to \mathbb{R}^{|X|-1}$ such that for any $v \in V \setminus X$, $f(v) \in \mathrm{conv}(f(N(v)))$.

A convex embedding is in general position if the set $f(V)$ of the points is in general position.

Proof of 3-connected with a triangle

Theorem (Linial, Lovász, and Wigderson 1988). Let G be a graph on n vertices. The following two conditions are equivalent:

1. G is k-connected
2. For every $X \subset V$ with $|X| = k$, G has a convex X-embedding in general position.

Proof.

- Assign to every edge $\{u, v\} \in E$ a positive *elasticity coefficient* c_{uv}
- $\forall v_i \in X$, let $f(v_i)$ arbitrary in $\mathbb{R}^{k-1}$ so that $f(X)$ is in general position
- For almost any set of elasticity coefficients, the embedding f that minimizes the potential function P provides a convex X-embedding in general position

$$P = \sum_{\{u,v\} \in E} c_{uv} ||f(u) - f(v)||^2$$

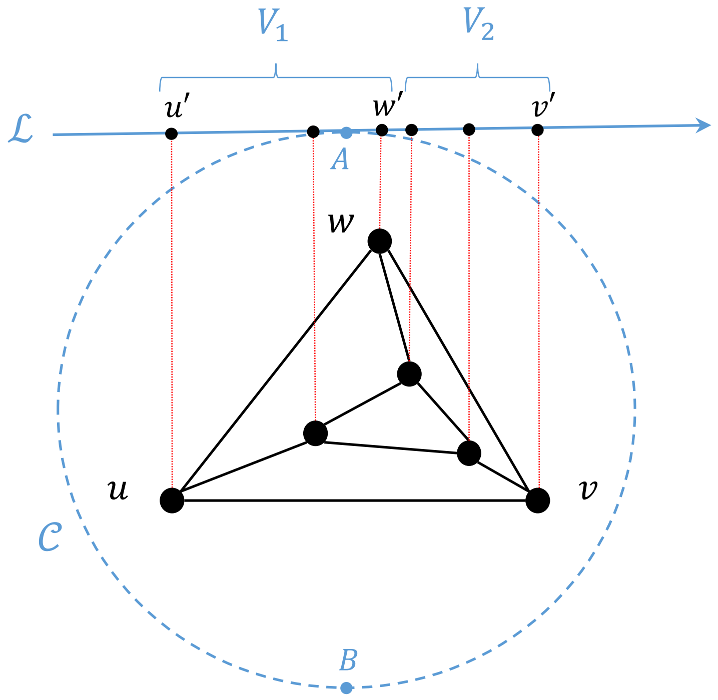
- The embedding can be computed as:

$$f(v) = \frac{1}{c_v} \sum_{u \in N(v)} c_{uv} f(u), \forall v \in V \backslash \mathrm{X}$$

in which $c_v = \sum_{u \in N(v)} c_{uv}$

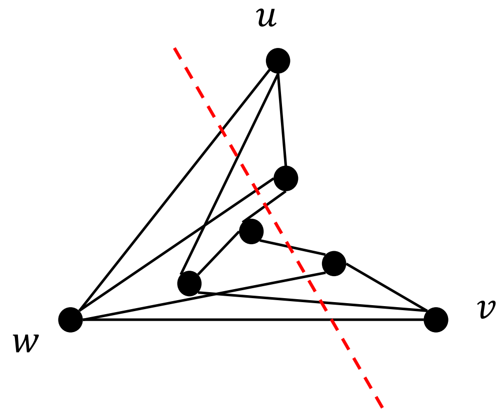
Proof of 3-connected with a triangle

- Assume u, w, v form a triangle in G
- $|V_1| = |V_2| = |V|/2$
- For each line tangent to $\mathcal{C}$, $G[V_1]$ and $G[V_2]$ are connected
- Move $\mathcal{L}$ from being tangent at point A to B
- As $\mathcal{L}$ moves, $p(V_i)$ changes at most by ± 2 or 0
- Somewhere in the middle we get $p(V_1) = p(V_2) = 0$

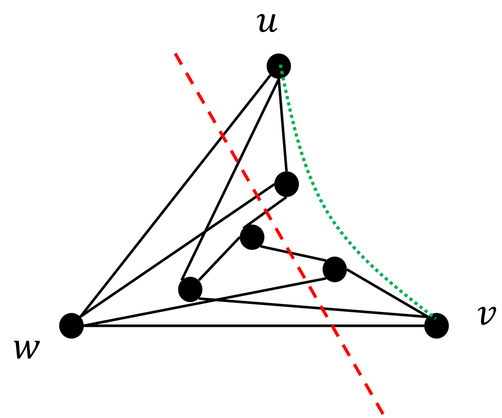


3-connected and triangle-free

- The previous proof does not work



- Find an embedding such that if u, v are in the same side, then so does the path that connects them

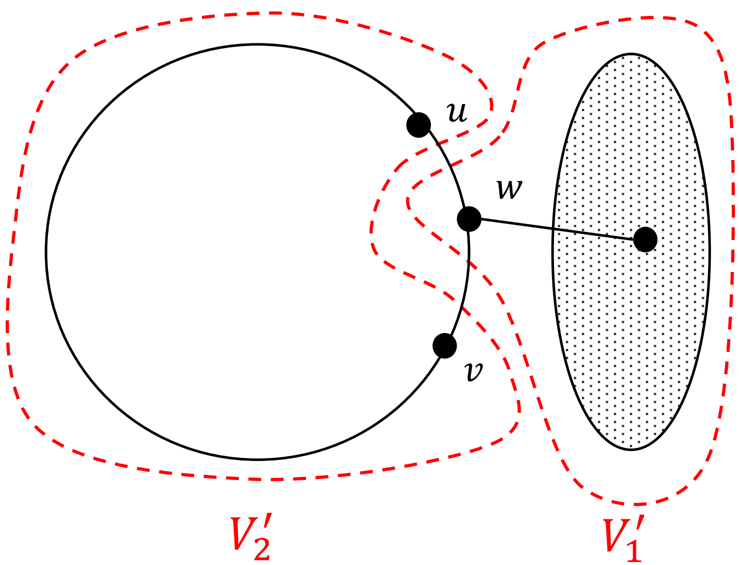


Lemma. If G *is 3-connected, then there exist a set* $\{u, v, w\} \in V$ and a partition of V into (V_1', V_2') such that:

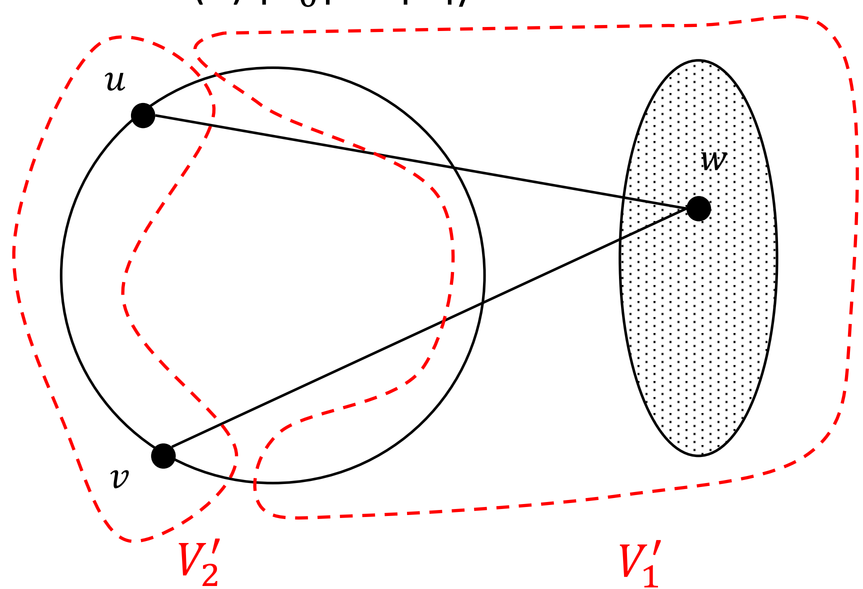
1. $G[V_1']$ and $G[V_2']$ are connected
2. $w \in V_1'$, $u, v \in V_2'$, and u, v are not cutpoints of $G[V_2']$
3. $\{u, w\}, \{v, w\} \in E$
4. $|V_2'| \leq |V|/2$

Proof. Consider a non-separating induced cycle in G (Tutte)

(a) $|C_0| \leq |\mathrm{V}|/2+1$



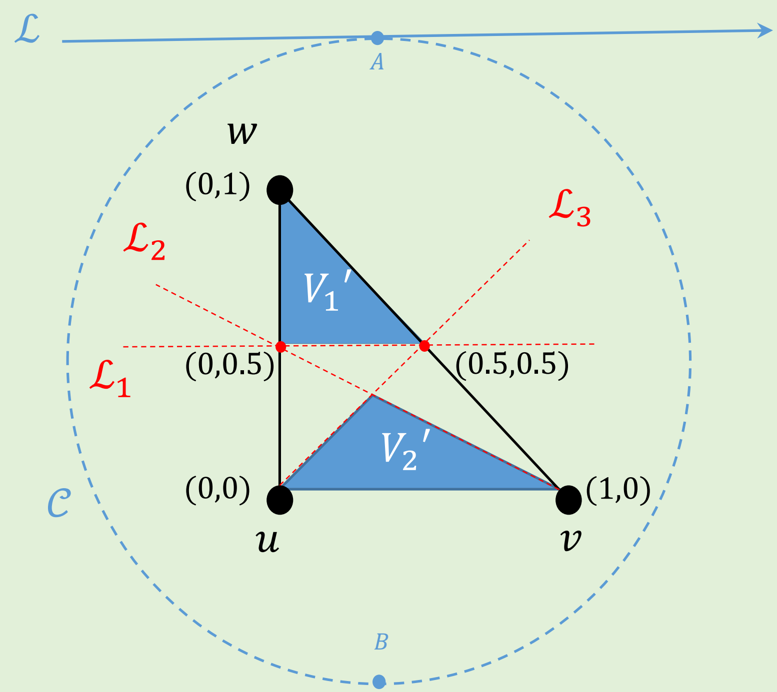
(b) $|C_0| > |\mathrm{V}|/2+1$



Lemma. Given a partion (V_1', V_2') of a 3-connected graph G with following properties:

1. $G[V_1']$ and $G[V_2']$ are connected
2. $w \in V_1'$, $u, v \in V_2'$, and u, v are not cutpoints of $G[V_2']$

G has a convex X-embedding in general position with mapping $f: V \rightarrow \mathbb{R}^2$ as below:



3-connected and triangle-free

Theorem. If G is a 3-connected graph, then

1. If $V \equiv 0 (\mathrm{mod}\ 4)$, then there exists a solution to the DBCP problem with $p(V_1) = p(V_2) = 0$ and $|V_1| = |V_2|$.
2. If $V \equiv 2 (\mathrm{mod}\ 4)$, then there exists a solution to the DBCP problem with $p(V_1) = p(V_2) = 0$ and $|V_1| = |V_2| + 2$.

Moreover, this solution can be find in polynomial time.

- Results can be generalized to arbitrary supply/demand values:

Corollary. If G is a 3-connected graph, then the DBCP problem has solution for $c_p = \max_{i \in V} |p(i)|$ and $c_s = 1$.

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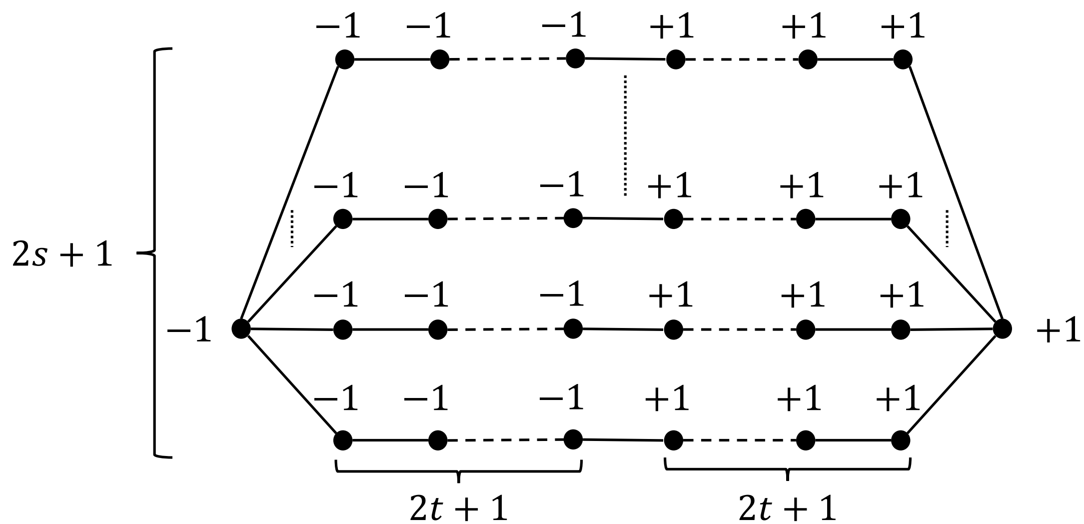
Extreme Cases



Observation. If $c_p = 0$, then for any $c_s < |V|/2 - 1$, there exist a 2-connected graph such that DBCP problem does not have solution.

Proof. Set $t = 0$.

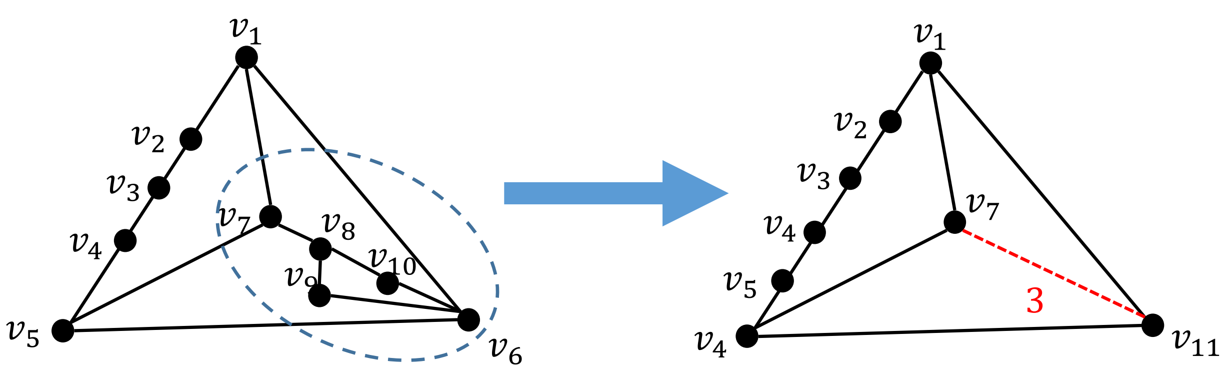
Observation. If $c_s = 1$, then for any $c_p < |V|/6$, there exist a 2-connected graph such that DBCP problem does not have solution.

Proof. Set $\mathrm{s} = 1$.

Graph Contraction

An induced subgraph H of a 2-connected graph G is *contractible*, if there is a separating pair $\{u, v\} \subset V$ such that $H = (V_H, E_H)$ is a connected component of $G[V \backslash \{u, v\}]$.

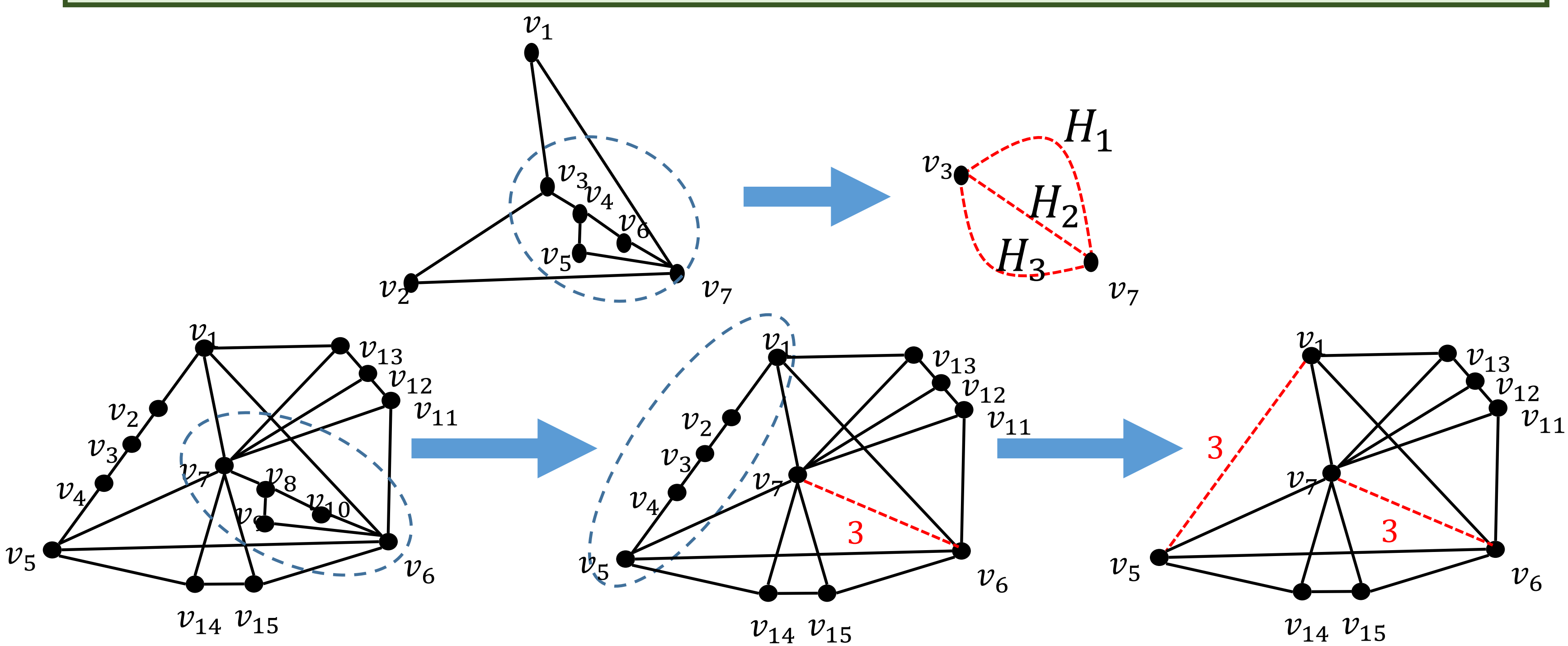
If we replace H by a weighted edge e' with $w(e') = |V_H|$ to obtain a smaller graph G', we say G is contracted to G'.



G G'

Lemma. In every 2-connected graph $G = (V, E)$, given an integer $q \geq 3$, one of the following cases holds:

1. There is a separation pair $\{u, v\} \subset V$ such that for each connected component H of $G[V \backslash \{u, v\}]$, $|V_H| < \lfloor \frac{(q-1)|V|}{q} \rfloor$
2. After a set of contractions, G can be transformed into a 3-connected graph $G^* = (V^*, G^*)$ such that for every e^*, $w(e^*) \leq \left\lceil \frac{|V|}{q} \right\rceil - 2$.

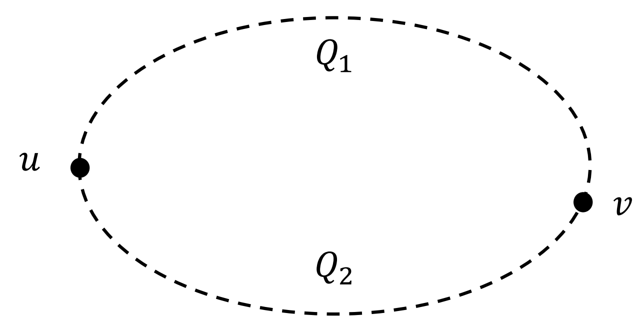


Lemma. Given a 2-connected graph G, and an integer $q \geq 3$, if G has a separation pair $\{u, v\} \subset V$ such that for every connected component $H_i = (V_{H_i}, E_{H_i})$ of $G[V \backslash \{u, v\}]$, $|V_{H_i}| < \lfloor \frac{(q-1)|V|}{q} \rfloor$, then the DBCP problem has a solution for $c_p = 1, c_s = q - 1$.

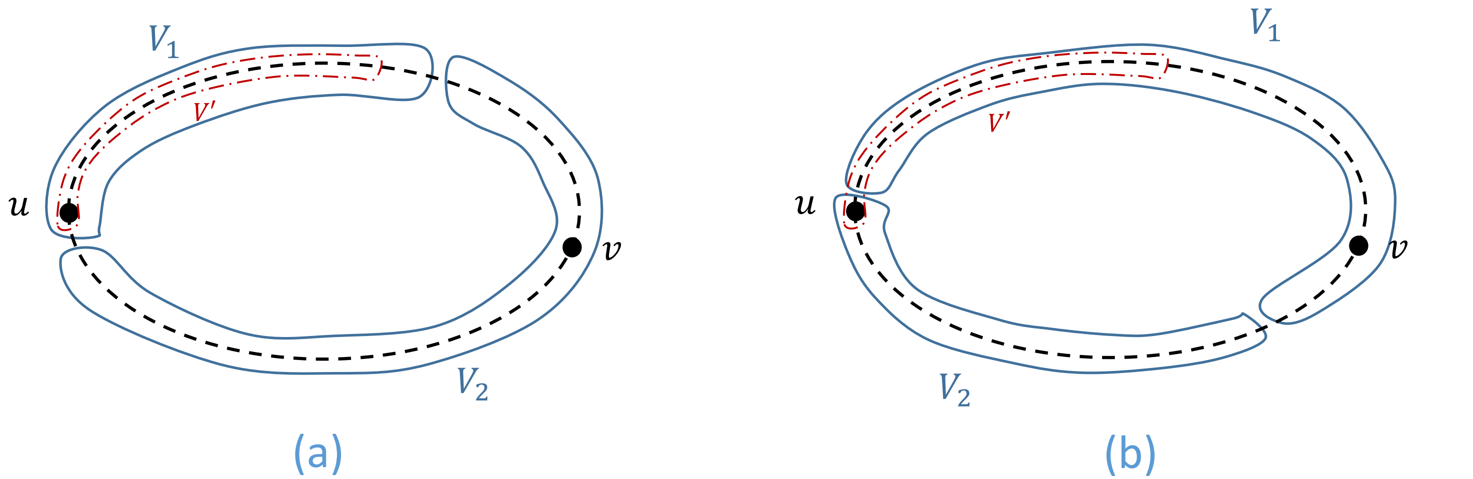
A *pseudo-path* between nodes u and v in $G = (V, E)$, is a sequence of nodes $v_1, v_2, \ldots, v_t$ such that if $v_0 := u$ and $v_{t+1} := v$ then for any $1 \leq i \leq t$, v_i has neighbors v_j and v_k such that $j < i < k$.

Proof. Each H_i can be shown by a pseudo path between u and v.
Divide pseudo-paths into two sets S_1 and S_2 such that:
$\sum_{P_j \in S_i} |P_j| \geq \left\lceil \frac{|V|}{q} \right\rceil - 1$

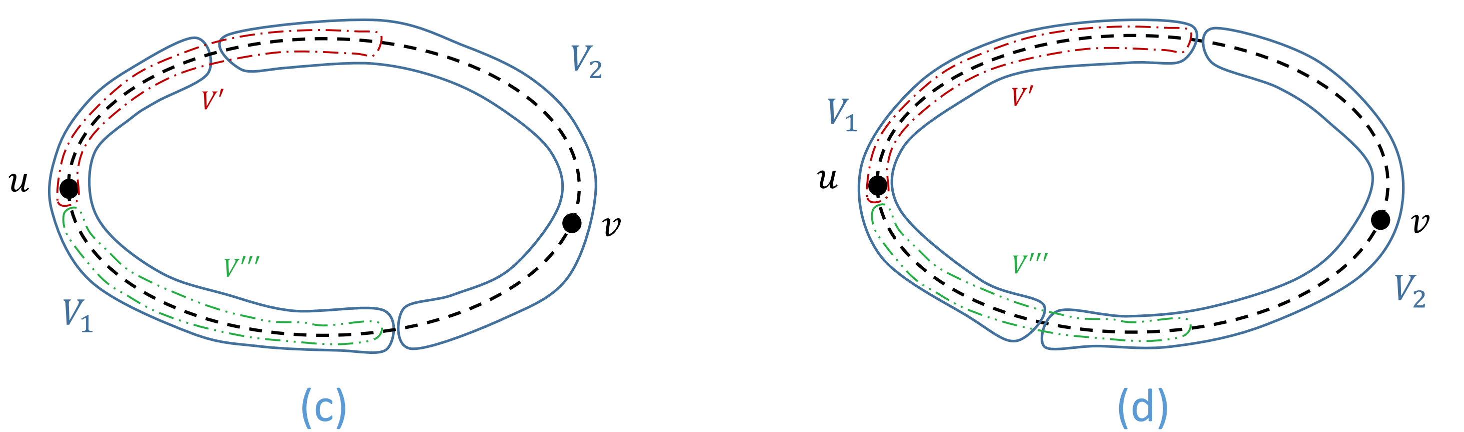
So G can be shown as follows such that $Q_1, Q_2 \geq \left\lceil \frac{|V|}{q} \right\rceil - 1$.



Proof. $|V'| = \frac{|V|}{q}$ *and assume* $p(V') \geq 0$.

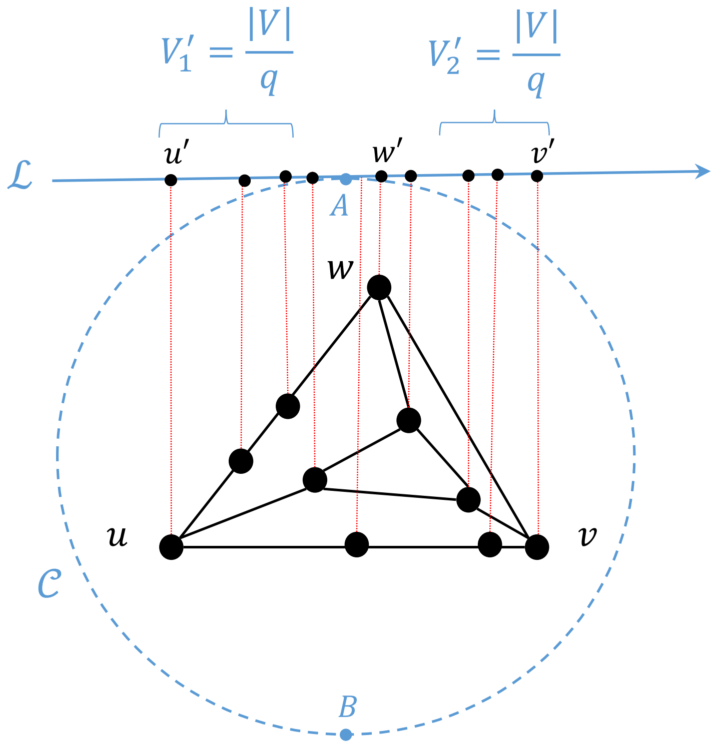


$|V'''| = \frac{|V|}{q}$ *and* $p(V''') < 0$.



Lemma. Given an integer $q \geq 4$, if after a set of contractions, G can be contracted into a 3-connected graph $G^* = (V^*, G^*)$ such that for every e^*, $w(e^*) \leq \left\lceil \frac{|V|}{q} \right\rceil - 2$. Then the DBCP problem has a solution for $c_p = 1, c_s = q - 1$.

- Some care is needed to carry out the argument of the 3-connected case for the contracted graph
- As we move $\mathcal{L}$, at some point $p(V_1')p(V_2') \geq 0$.



2-connected

Theorem. If G is 2-connected, then the DBCP problem has a solution for $c_p = 1$ and $c_s = 3$. Moreover, this solution can be found in polynomial time.

- Recently showed that for $c_p = 1$ and $c_s = 2$ has a solution.

Corollary. If G is 2-connected, then the DBCP problem with arbitrary weights has a solution for $c_p = \max_{j \in V} |p(j)|$ and $c_s = 3$.

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Graphs with Two Types of Nodes

Corollary. Given a 3-connected graph G with nodes colored either red ($R \subseteq V$) or blue ($B \subseteq V$). There is always a partition (V_1, V_2) of V such that

1. $G[V_1]$ and $G[V_2]$ are connected
2. $|V_1| = |V_2|$
3. $|R \cap V_1| = |R \cap V_2|$ and $|B \cap V_1| = |B \cap V_2|$ (assuming $|R|$ and $|B|$ are both even)

Corollary. Given a 2-connected graph G with nodes colored either red ($R \subseteq V$) or blue ($B \subseteq V$). There is always a partition (V_1, V_2) of V such that

1. $G[V_1]$ and $G[V_2]$ are connected
2. $|V_1|, |V_2| \geq |V|/4$
3. The ratio of red to blue nodes in each side V_i differs from $|R|/|B|$ by $O(1/n)$.

Thank You!